

# EE 261 The Fourier Transform and its Applications

## Fall 2007

### Problem Set Three    Due Wednesday, October 17

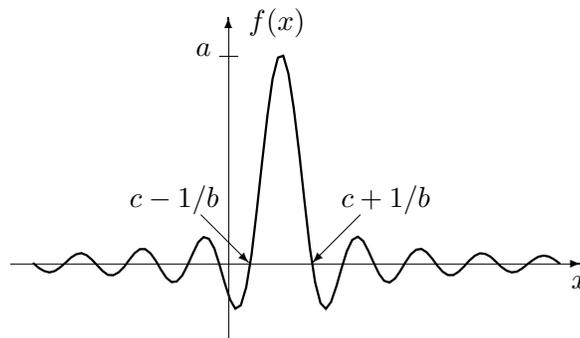
1. (5 points) *Still another reciprocal relationship* The *equivalent width* of a signal  $f(t)$ , with  $f(0) \neq 0$ , is the width of a rectangle having height  $f(0)$  and area the same as under the graph of  $f(t)$ . Thus

$$W_f = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(t) dt.$$

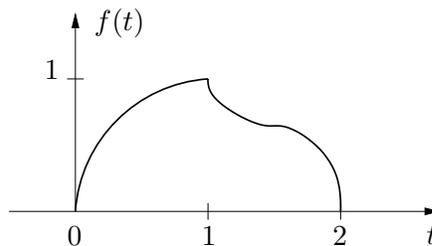
This is a measure for how spread out a signal is.

Show that  $W_f W_{\mathcal{F}f} = 1$ . Thus, the equivalent widths of a signal and its Fourier transform are reciprocal.

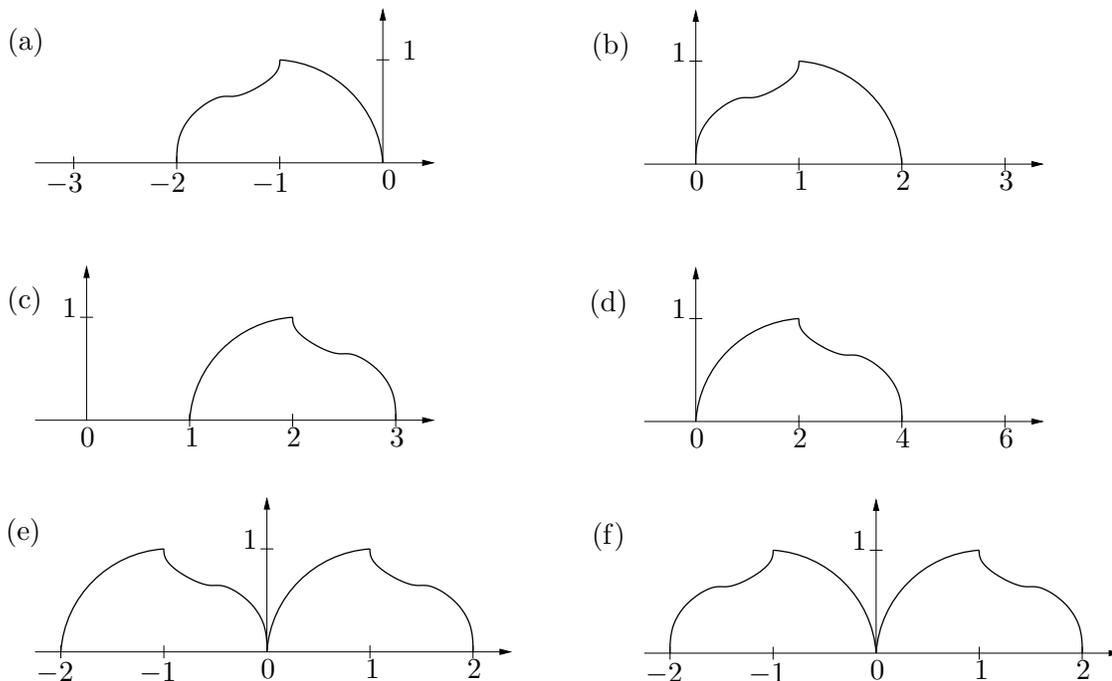
2. (10 points) Find the Fourier transforms of the function shown in the graph (a shifted sinc)



3. (5 points each) The figures below show a signal  $f(t)$  and six other signals derived from  $f(t)$ . Note the scales on the axes.



Suppose  $f(t)$  has Fourier transform  $F(s)$ . Express the Fourier transforms of the other six signals in terms of  $F(s)$ .



4. (35 points) *Some practice with convolution*

In this problem, we want you to have some practice with handling convolution and integration. So, for parts (a) to (c), explicitly evaluate the convolution integral. Also verify your results by applying the convolution theorem for Fourier transforms.

- (a) What is  $\Pi_a * \Pi_a$ ?
- (b) Let  $f(x) = e^{-|x|}$ ,  $-\infty < x < \infty$ . Find  $(f * f)(x)$ .
- (c) Let  $g(x) = e^{-\pi x^2}$ ,  $-\infty < x < \infty$ . Show that  $(g * g)(x) = \frac{1}{\sqrt{2}} e^{-\pi x^2/2}$ .
- (d) From the result in part(c), deduce the result of the  $n$ -fold convolution of  $g$ , i.e.,  $g * g * \dots * g$  (with  $n$  factors of  $g$ ).

5. (30 points) *Convolution: Reversals, Shifts and Stretches* Let  $f(t)$  and  $g(t)$  be signals.

- (a) If both  $f(t)$  and  $g(t)$  are reversed, what happens to their convolution? If one of  $f(t)$  and  $g(t)$  is reversed what happens to their convolution?

Define the shift operator  $\tau_b f$  and the stretch operator  $\sigma_a f$  by

$$\tau_b f(t) = f(t - b), \quad \sigma_a f(t) = f(at).$$

- (b) Show that

$$(\tau_b f) * g = \tau_b(f * g) = f * (\tau_b g).$$

Write out in words what this says. Use this result to deduce that if either  $f$  or  $g$  is periodic of period  $T$  then  $f * g$  is periodic of period  $T$ . (However, see Problem 6 below!)

- (c) Show that

$$(\sigma_a f) * g = \frac{1}{|a|} \sigma_a(f * (\sigma_{1/a} g)), \quad (\sigma_a f) * (\sigma_a g) = \frac{1}{|a|} \sigma_a(f * g).$$

Write out in words what these identities say.

6. (10 points) Rajiv and Lykomidis are arguing about convolution over dinner one night:

Rajiv: You know, convolution really is a remarkable operation, the way it imparts properties of one function onto the convolution with another. Take periodicity – if  $f(t)$  is periodic then  $(f * g)(t)$  is periodic with the same period. I think that was a homework problem we gave the class.

Lykomidis: There's a problem with that statement. You want to say that if  $f(t)$  is a periodic function of period  $T$  then  $(f * g)(t)$  is also periodic of period  $T$ .

Rajiv: Right.

Lykomidis: What if  $g(t)$  is also periodic, say of period  $R$ ? Then doesn't  $(f * g)(t)$  have two periods,  $T$  and  $R$ ?

Rajiv: I suppose so.

Lykomidis: But wouldn't this lead right to a contradiction? I mean, for example, you can't have a function with two periods, can you?

The conversation continues. They are joined by Thomas:

Rajiv and Lykomidis: We think we've found a fundamental contradiction in mathematics.

Thomas: Why don't you look at a simple, special case first. What happens if you convolve  $\sin 2\pi t$  with itself?

Rajiv: OK, both functions have period 1 so for the convolution you get a function that's periodic of period 1, no problem.

Lykomidis: No, you don't. Something goes wrong.

What's going on? With whom do you agree and why?

7. (25 points)

*Probability Distributions, Convolution and MATLAB.* You have three six-sided dice, one white, one red and one black. The white one is fair but the red and the black are not, in particular:

	Prob(1)	Prob(2)	Prob(3)	Prob(4)	Prob(5)	Prob(6)
white	1/6	1/6	1/6	1/6	1/6	1/6
red	1/12	1/12	2/12	2/12	3/12	3/12
black	4/12	3/12	2/12	1/12	1/12	1/12

(a) (10 points) You roll each die once. What is the probability that the sum of all the three rolls is equal to 7 or 8 or 12. (Hint: You may want to use matlab)

(b) (5 points) You repeat rolling all three dice many times. (Each time you roll each die once.) Let  $X_i$ ,  $Y_i$  and  $Z_i$  be the random variable denoting the  $i$ -th roll of the white, red and black die correspondingly. As the number of trials  $N$  increases, what do you think the empirical average of the numbers

$$\frac{1}{3N} \sum_{i=1}^N (X_i + Y_i + Z_i)$$

should be.

- (c) (10 points) Use MATLAB to plot the distribution of the empirical average, for  $N = 2, 10, 100$  and  $1000$ .