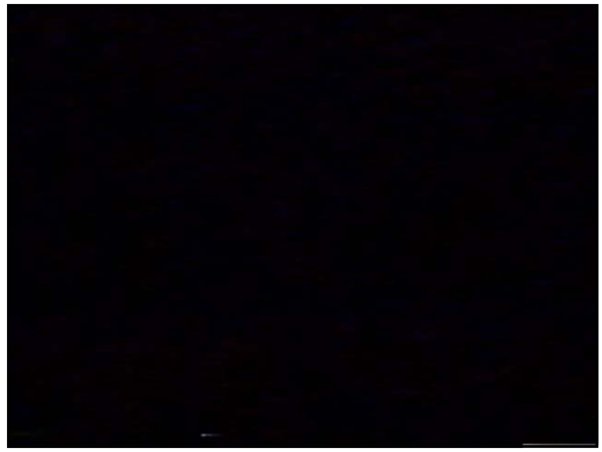
# Movie Segment

Locomotion Gates with Polypod,
Mark Yim, Stanford University,
ICRA 1994 video proceedings

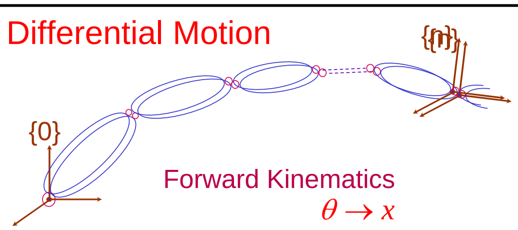


# Instantaneous Kinematics

# Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

## Differential Motion




Forward Kinematics

$\theta \rightarrow x$

Instantaneous Kinematics

$\theta + \delta\theta \rightarrow x + \delta x$

Relationship: $\boxed{\delta\theta \leftrightarrow \delta x}$

$\boxed{\dot{\theta} \leftrightarrow \dot{x}}$ — Linear Velocity, Angular Velocity

## Joint Coordinates

$$coordinate - i : \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$$

Joint coordinate-i: $\boxed{q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i}$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $\boxed{q = (q_1 q_2 \ldots q_n)^T}$

## Jacobians: Direct Differentiation

$$x = f(q); \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} f_1(q) \\ f_2(q) \\ \vdots \\ f_m(q) \end{pmatrix}$$

$$\delta x_1 = \frac{\partial f_1}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_1}{\partial q_n}\delta q_n$$

$$\vdots$$

$$\delta x_m = \frac{\partial f_m}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_m}{\partial q_n}\delta q_n$$

$$\delta x = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} . \delta q$$

$$\boxed{\delta x_{(m\times 1)} = J_{(m\times n)}(q)\,\delta q_{(n\times 1)}}$$

## Jacobian

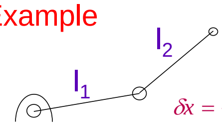
$$\delta x_{(m\times 1)} = J_{(m\times n)}(q)\,\delta q_{(n\times 1)}$$

$$\dot{x}_{(m\times 1)} = J_{(m\times n)}(q)\,\dot{q}_{(n\times 1)}$$

where

$$J_{ij}(q) = \frac{\partial}{\partial q_j} f_i(q)$$

## Example




$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

$$\delta x = -(l_1 s_1 + l_2 s_{12})\delta\theta_1 - l_2 s_{12}\delta\theta_2$$

$$\delta y = (l_1 c_1 + l_2 c_{12})\delta\theta_1 + l_2 c_{12}\delta\theta_2$$

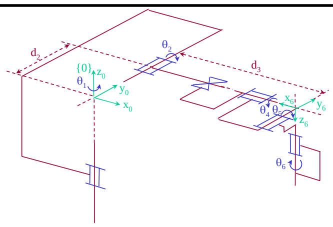
$$\delta X = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix}$$

$$\boxed{\delta x = J(\theta)\delta\theta}$$

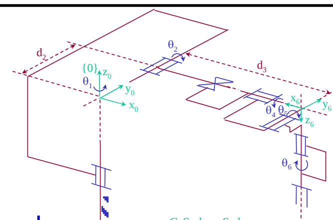
$$\dot{x} = J(\theta)\dot{\theta}$$

$$J \equiv \begin{pmatrix} \frac{\partial x}{\partial\theta_1} & \frac{\partial x}{\partial\theta_2} \\ \frac{\partial y}{\partial\theta_1} & \frac{\partial y}{\partial\theta_2} \end{pmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix}$$

## *Stanford Scheinman* Arm




| i | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | -90 | 0 | $d_2$ | $\theta_2$ |
| 3 | 90 | 0 | $d_3$ | 0 |
| 4 | 0 | 0 | 0 | $\theta_4$ |
| 5 | -90 | 0 | 0 | $\theta_5$ |
| 6 | 90 | 0 | 0 | $\theta_6$ |




$$x = \begin{pmatrix} x_P \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{vmatrix} C_1 S_2 d_3 - S_1 d_2 \\ S_1 S_2 d_3 + C_1 d_2 \\ C_2 d_3 \\ C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{vmatrix}$$

## Stanford Scheinman Arm

Position

$$x_p = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

$$\dot{x}_p = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

$$\dot{x}_{p(3\times1)} = J_{x_p(3\times6)}(q)\,\dot{q}_{(6\times1)}$$

Linear Velocity **v**

## Orientation: Direction Cosines

$$x_R = \begin{bmatrix} r_1(q) \\ r_2(q) \\ r_3(q) \end{bmatrix}$$

$$\dot{x}_R = J_{x_R}(q)\dot{q}$$

$$\dot{x}_R = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix}_{(9x1)} = \begin{pmatrix} \frac{\partial r_1}{\partial q_1} & \cdots & \frac{\partial r_1}{\partial q_6} \\ \frac{\partial r_2}{\partial q_1} & \cdots & \frac{\partial r_2}{\partial q_6} \\ \frac{\partial r_3}{\partial q_1} & \cdots & \frac{\partial r_3}{\partial q_6} \end{pmatrix}_{(9x6)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_6 \end{pmatrix}_{(6x1)}$$

$$x_R = \begin{vmatrix} C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{vmatrix}$$

$$\dot{x}_R = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix}_{(9x1)} = \begin{pmatrix} \frac{\partial r_1}{\partial q_1} & \cdots & \frac{\partial r_1}{\partial q_6} \\ \frac{\partial r_2}{\partial q_1} & \cdots & \frac{\partial r_2}{\partial q_6} \\ \frac{\partial r_3}{\partial q_1} & \cdots & \frac{\partial r_3}{\partial q_6} \end{pmatrix}_{(9x6)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_6 \end{pmatrix}_{(6x1)}$$

## Representations

$$X = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$$

$x_P$:
- Cartesian
- Spherical
- Cylindrical
- ....

$x_R$:
- Euler Angles
- Direction Cosines
- Euler Parameters

## Jacobian for X

$$\dot{x}_P = J_{X_P}(q)\dot{q}$$
$$\dot{x}_R = J_{X_R}(q)\dot{q}$$

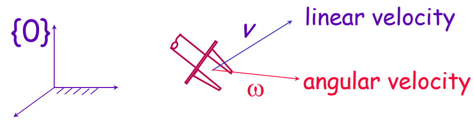
$$\begin{pmatrix} \dot{x}_P \\ \dot{x}_R \end{pmatrix} = \begin{pmatrix} J_{X_P}(q) \\ J_{X_R}(q) \end{pmatrix}\dot{q}$$

Cartesian & Direction Cosines

$$\dot{X}_{(12x1)} = J_X(q)_{(12x6)}\,\dot{q}_{(6x1)}$$

The Jacobian is dependent on the representation

## Basic Jacobian




$$\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6x1)} = J_0(q)_{(6xn)}\,\dot{q}_{(nx1)}$$

$$\dot{x}_P = E_P(x_P)v$$
$$\dot{x}_R = E_R(x_R)\omega$$

## Examples

$\star\ x_R = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}; E_R(x_R) = \begin{pmatrix} -\frac{s\alpha.c\beta}{s\beta} & \frac{c\alpha.c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{pmatrix}$

$\star\ x_P = \begin{pmatrix}x\\ y\\ z\end{pmatrix}; E_P(x_P) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

## Jacobian for X

Given a representation $x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$

$$\dot{x} = J_x(q)\,\dot{q}$$

$$J_x(q) = E(x)\ J_0(q)$$

Basic Jacobian $\begin{pmatrix} v \\ w \end{pmatrix} = J_0(q)\ \dot{q}$

## Jacobian and Basic Jacobian

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q).\dot{q}$$

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} J_v \\ J_\omega \end{pmatrix}.\dot{q}$$

$$\begin{cases} v = J_v.\dot{q} \\ \omega = J_\omega.\dot{q} \end{cases}$$

$$\dot{x}_P = E_P.v \Rightarrow \dot{x}_P = (E_P.J_v)\dot{q}$$

$$\dot{x}_R = E_R.\omega \Rightarrow \dot{x}_R = (E_R.J_\omega)\dot{q}$$

$$\begin{cases} J_{X_P} = E_P.J_v \\ J_{X_R} = E_R.J_\omega \end{cases}$$

$$J = \begin{pmatrix} J_{XP} \\ J_{XR} \end{pmatrix} = \left(\begin{array}{c|c} E_P & 0 \\ \hline 0 & E_R \end{array}\right) \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

$$J(q) \quad = \quad E(X) \quad J_0(q)$$

$$\begin{pmatrix} v \\ w \end{pmatrix} \quad = \quad J_0(q)\ \dot{q}$$

With **Cartesian Coordinates**

$E_P = I_3\ ;\ J_{XP} = J_v\ ;$ and $E = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & E_R \end{array}\right)$

## Position Representations

Cartesian Coordinates $(x, y, z)$

$$E_P(X) = I_3$$

Cylindrical Coordinates $(\rho, \theta, z)$

Using $(x\ \ y\ \ z)^T = (\rho\cos\theta\ \ \ \rho\sin\theta\ \ \ z)^T$

$$E_P(X) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates $(\rho, \theta, \phi)$

Using

$(x\ \ y\ \ z)^T = (\rho\cos\theta\sin\phi\ \ \ \rho\sin\theta\sin\phi\ \ \ \rho\cos\theta)^T$

$$E_P(X) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & -\sin\phi/\rho \end{pmatrix}$$

## Euler Angles

$$x_R = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}; E_R(x_R) = \begin{pmatrix} -\frac{s\alpha . c\beta}{s\beta} & \frac{c\alpha . c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{pmatrix}$$

Singularity of the representation for $\beta = k\pi$

## Jacobian for X

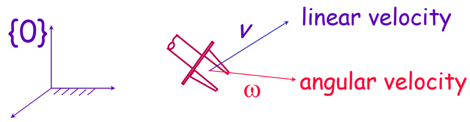
Given a representation $x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$

$$\dot{x} = J_x(q)\,\dot{q}$$

$$J_x(q) = E(x)\; J_0(q)$$

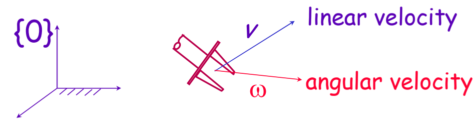
Basic Jacobian $\begin{pmatrix} v \\ w \end{pmatrix} = J_0(q)\;\dot{q}$

## Jacobian

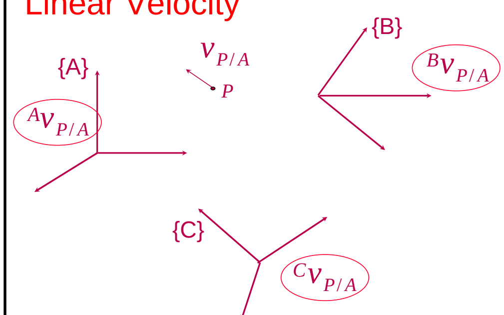



$$\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6x1)} = J(q)_{(6xn)}\,\dot{q}_{(nx1)}$$

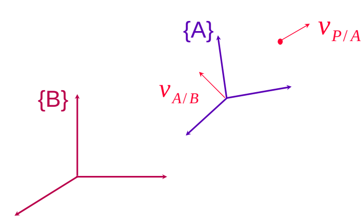
## Linear & Angular Velocities



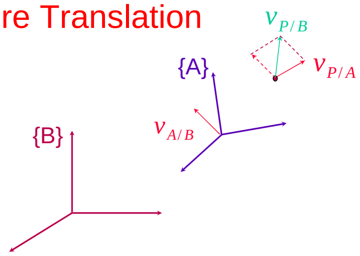

## Linear Velocity




## Pure Translation

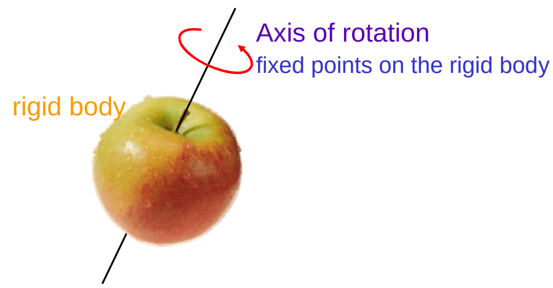


## Pure Translation




$$v_{P/B} = v_{A/B} + v_{P/A}$$

## Rotational Motion

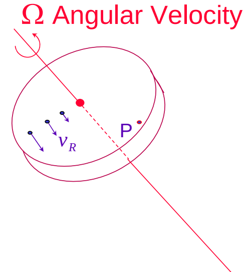



## Rotational Motion

$\Omega$ Angular Velocity

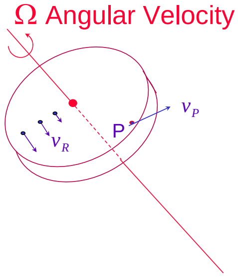
## Rotational Motion

$\Omega$ Angular Velocity




## Rotational Motion

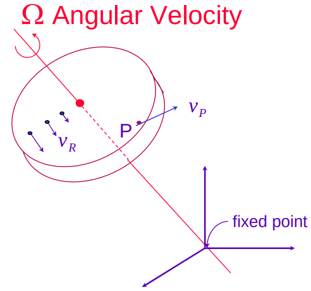
$\Omega$ Angular Velocity




$$v_P = ?$$

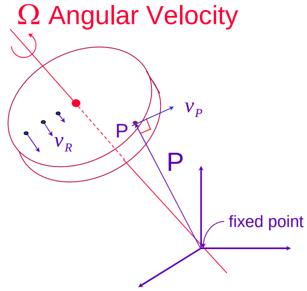
## Rotational Motion

$\Omega$ Angular Velocity



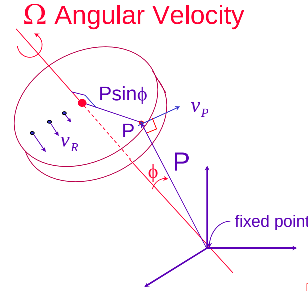
## Rotational Motion

$\Omega$ Angular Velocity




## Rotational Motion

$\Omega$ Angular Velocity




$v_P$ is proportional to:

- $||\Omega||$
- $||P\sin\phi||$

and

- $v_P \perp \Omega$
- $v_P \perp P$

$$v_P = \Omega \times P$$

## Cross Product Operator

$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$c = a \times b \Rightarrow c = \hat{a}b$$

vectors ⇨ matrices

$a \times \Rightarrow \hat{a}$ : a skew-symmetric matrix

$$c = \hat{a}b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$c = \hat{a}b$$

## Cross Product Operator

$$v_P = \Omega \times P \Rightarrow v_P = \hat{\Omega}P$$

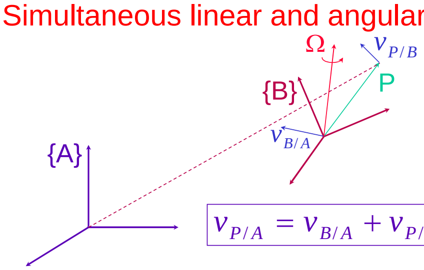
$\Omega \times \Rightarrow \hat{\Omega}$ : a skew-symmetric matrix

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}; P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$v_P = \hat{\Omega}P = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} . \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$v_P = \hat{\Omega}P$$

## Simultaneous linear and angular motion



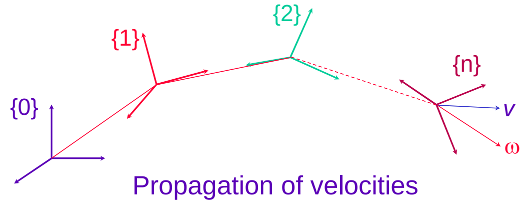

$$v_{P/A} = v_{B/A} + v_{P/B} + \Omega \times P_B$$

$${}^{A}v_{P/A} = {}^{A}v_{B/A} + {}^{A}_{B}R.{}^{B}v_{P/B} + {}^{A}\Omega_B \times {}^{A}_{B}R.{}^{B}P_B$$

## Movie Segment

Beach Volleyball, Toshiba,
ICRA 1999 video proceedings

## Spatial Mechanisms

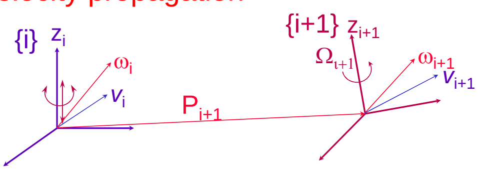



Propagation of velocities

$\dot{x}$ < $v$ : linear velocity; ω : angular velocity

$$\dot{x} = J(\theta).\dot{\theta}$$

## Velocity propagation




Linear $\quad v_{i+1} = v_i + \omega_i \times P_{i+1} + \dot{d}_{i+1}.Z_{i+1}$

Angular $\quad \omega_{i+1} = \omega_i + \Omega_{i+1}$

$\Omega_{i+1} = \dot{\theta}_{i+1}.Z_{i+1}$

## Velocity propagation

Joint 1

$v_1$ and $\omega_1$ in frame {1}

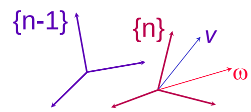
Joint i+1

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R.{}^{i}\omega_i + \dot{\theta}_{i+1}.{}^{i+1}Z_{i+1}$$

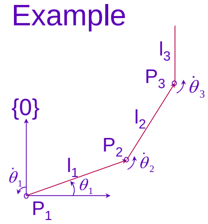
$$^{i+1}v_{i+1} = {}^{i+1}_{i}R.({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1}.{}^{i+1}Z_{i+1}$$

$\Rightarrow$ ${}^{n}\omega_n$ and ${}^{n}v_n$

$$\begin{pmatrix} {}^0v_n \\ {}^0\omega_n \end{pmatrix} = \begin{pmatrix} {}^0_nR & 0 \\ 0 & {}^0_nR \end{pmatrix}.\begin{pmatrix} {}^nv_n \\ {}^n\omega_n \end{pmatrix}$$




## Example




$$v_{i+1} = v_i + \omega_i \times P_{i+1}$$

- $v_{P_1} = 0 \qquad {}^0\omega_1 = \dot{\theta}_1.{}^0Z_1$
- $v_{P_2} = v_{P_1} + \omega_1 \times P_2$
- $v_{P_3} = v_{P_2} + \omega_2 \times P_3$

$$^0v_{P_2} = 0 + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\begin{bmatrix} l_1.c_1 \\ l_1.s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix}.\dot{\theta}_1$$

$$^0v_{P_3} = {}^0v_{P_2} + {}^0\omega_2 \times {}^0P_3$$

$$^0v_{P_3} = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix}.\dot{\theta}_1 + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.(\dot{\theta}_1 + \dot{\theta}_2).{}^0P_3$$

$${}^0P_3 \leftarrow \begin{bmatrix} l_2.c_{12} \\ l_2.s_{12} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix}.\dot{\theta}_1 + \begin{bmatrix} -l_2.s_{12} \\ l_2.c_{12} \\ 0 \end{bmatrix}.(\dot{\theta}_1 + \dot{\theta}_2)$$

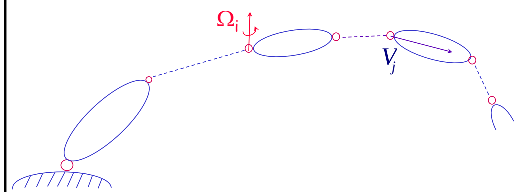
$$^0\omega_3 = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3).{}^0Z_0$$

$$
{}^{0}v_{P_3} = \underbrace{\begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J_v} . \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix}
$$

$$
{}^{0}\omega_3 = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{J_\omega} . \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix}
$$

$$
\begin{pmatrix} v \\ \omega \end{pmatrix} = J . \begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{pmatrix}
$$

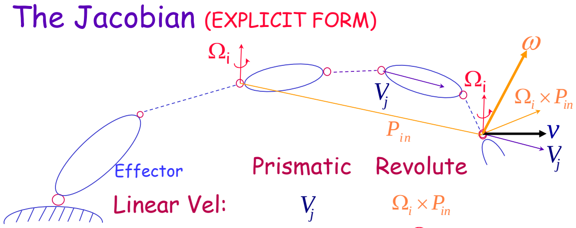
## The Jacobian (EXPLICIT FORM)




Revolute Joint $\Omega_i = Z_i \dot q_i$

Prismatic Joint $V_i = Z_i \dot q_i$

## The Jacobian (EXPLICIT FORM)




| | Prismatic | Revolute |
|---|---|---|
| Linear Vel: | $V_j$ | $\Omega_i \times P_{in}$ |
| Angular Vel: | none | $\Omega_i$ |

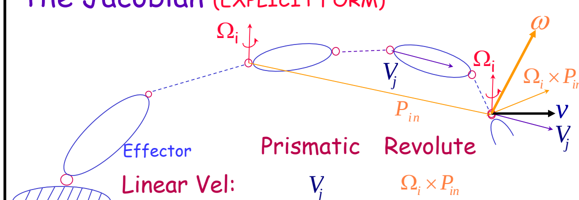
Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i \, V_i + \bar\epsilon_i \, (\Omega_i \times P_{in})] \quad \Leftarrow \quad V_i = Z_i \dot q_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} \bar\epsilon_i \, \Omega_i \quad \Leftarrow \quad \Omega_i = Z_i \dot q_i$$

## The Jacobian (EXPLICIT FORM)




| | Prismatic | Revolute |
|---|---|---|
| Linear Vel: | $V_j$ | $\Omega_i \times P_{in}$ |
| Angular Vel: | none | $\Omega_i$ |

Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i \, Z_i + \bar\epsilon_i \, (Z_i \times P_{in})] \dot q_i \quad \Leftarrow \quad V_i = Z_i \dot q_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} (\bar\epsilon_i \, Z_i) \, \dot q_i \quad \Leftarrow \quad \Omega_i = Z_i \dot q_i$$

$$v = [\epsilon_1 \, Z_1 + \bar\epsilon_1 \, (Z_1 \times P_{1n})] \dot q_1 + \cdots$$
$$+ [\epsilon_{n-1} \, Z_{n-1} + \bar\epsilon_{n-1} \, (Z_{n-1} \times P_{(n-1)n})] \dot q_{n-1} + \epsilon_n \, Z_n \dot q_n$$

$$v = \left[ \epsilon_1 \, Z_1 + \bar\epsilon_1 \, (Z_1 \times P_{1n}) \quad \epsilon_2 \, Z_2 + \bar\epsilon_2 \, (Z_2 \times P_{2n}) \quad \cdots \right] \begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \vdots \\ \dot q_n \end{bmatrix}$$

$$\boxed{v = J_v \dot q}$$

$$\omega = \bar\epsilon_1 \, Z_1 \dot q_1 + \bar\epsilon_2 \, Z_2 \dot q_2 + \cdots + \bar\epsilon_n \, Z_n \dot q_n$$

$$\omega = \left[ \bar\epsilon_1 \, Z_1 \quad \bar\epsilon_2 \, Z_2 \quad \cdots \quad \bar\epsilon_n \, Z_n \right] \begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \vdots \\ \dot q_n \end{bmatrix}$$

$$\boxed{\omega = J_\omega \dot q}$$

## The Jacobian

$$J = \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

### Matrix $J_v$ (direct differentiation)

$$v = \begin{pmatrix} \dot x \\ \dot y \\ \dot z \end{pmatrix} = \dot x_P = \frac{\partial x_P}{\partial q_1} . \dot q_1 + \frac{\partial x_P}{\partial q_2} . \dot q_2 + \cdots + \frac{\partial x_P}{\partial q_n} . \dot q_n$$

$$J_v = \begin{pmatrix} \dfrac{\partial x_P}{\partial q_1} & \dfrac{\partial x_P}{\partial q_2} & \cdots & \dfrac{\partial x_P}{\partial q_n} \end{pmatrix}$$

## Jacobian in a Frame

Vector Representation

$$J = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \\ \bar{\epsilon}_1 . Z_1 & \bar{\epsilon}_2 . Z_2 & \cdots & \bar{\epsilon}_n . Z_n \end{pmatrix}$$

In {0}

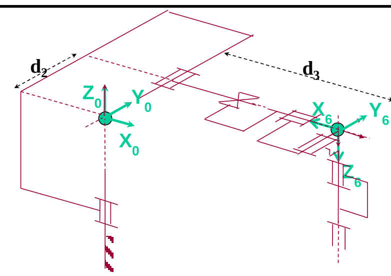
$$^0J = \begin{pmatrix} \frac{\partial\, ^0x_P}{\partial q_1} & \frac{\partial\, ^0x_P}{\partial q_2} & \cdots & \frac{\partial\, ^0x_P}{\partial q_n} \\ \bar{\epsilon}_1 .\, ^0Z_1 & \bar{\epsilon}_2 .\, ^0Z_2 & \cdots & \bar{\epsilon}_n .\, ^0Z_n \end{pmatrix}$$

## J in Frame {0}

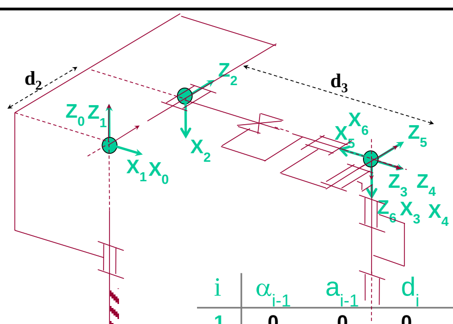
$$^0Z_i = {}^0_iR\; ^iZ_i; \quad ^iZ_i = Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$^0J = \begin{pmatrix} \frac{\partial}{\partial q_1}(^0x_P) & \frac{\partial}{\partial q_2}(^0x_P) & \cdots & \frac{\partial}{\partial q_n}(^0x_P) \\ \bar{\epsilon}_1 .({}^0_1R.Z) & \bar{\epsilon}_2 .({}^0_2R.Z) & \cdots & \bar{\epsilon}_n .({}^0_nR.Z) \end{pmatrix}$$

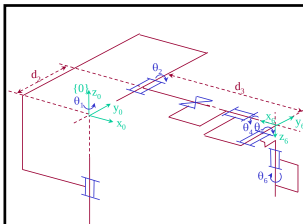
## Stanford Scheinman Arm




$$J = \left(\begin{array}{c|c|c|c|c|c} Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & 0 & 0 & 0 \\ \hline Z_1 & Z_2 & 0 & Z_4 & Z_5 & Z_6 \end{array}\right)$$




| i | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | -90 | 0 | $d_2$ | $\theta_2$ |
| 3 | 90 | 0 | $d_3$ | 0 |
| 4 | 0 | 0 | 0 | $\theta_4$ |
| 5 | -90 | 0 | 0 | $\theta_5$ |
| 6 | 90 | 0 | 0 | $\theta_6$ |



| i | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | -90 | 0 | $d_2$ | $\theta_2$ |
| 3 | 90 | 0 | $d_3$ | 0 |
| 4 | 0 | 0 | 0 | $\theta_4$ |
| 5 | -90 | 0 | 0 | $\theta_5$ |
| 6 | 90 | 0 | 0 | $\theta_6$ |

$${}^{i-1}_{\;\;i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics: ${}^0_NT = {}^0_1T\; {}^1_2T \ldots {}^{N-1}_{\;\;N}T$

## Stanford Scheinman Arm

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^4_5T = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^5_6T = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_2T = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 & -s_1d_2 \\ s_1c_2 & -s_1s_2 & c_1 & c_1d_2 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

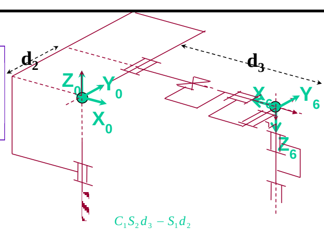
$$ {}^0_3T = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_4T = \begin{bmatrix} c_1c_2c_4 - s_1s_4 & -c_1c_2s_4 - s_1c_4 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2c_4 + c_1s_4 & -s_1c_2s_4 + c_1c_4 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2c_4 & s_2s_4 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_5T = \begin{bmatrix} X & X & -c_1c_2s_4 - s_1c_4 & c_1d_3s_2 - s_1d_2 \\ X & X & -s_1c_2s_4 + c_1c_4 & s_1d_3s_2 + c_1d_2 \\ X & X & s_2s_4 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_6T = \begin{bmatrix} X & X & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 & c_1d_3s_2 - s_1d_2 \\ X & X & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 & s_1d_3s_2 + c_1d_2 \\ X & X & -s_2c_4s_5 + c_5c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_6T = \begin{bmatrix} X & X & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 & c_1d_3s_2 - s_1d_2 \\ X & X & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 & s_1d_3s_2 + c_1d_2 \\ X & X & -s_2c_4s_5 + c_5c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$




$$ x = \begin{pmatrix} x_P \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{vmatrix} C_1S_2d_3 - S_1d_2 \\ S_1S_2d_3 + C_1d_2 \\ C_2d_3 \\ C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{vmatrix} $$

## Stanford Scheinman Arm Jacobian

$$ {}^0J = \begin{pmatrix} \frac{\partial^0 x_P}{\partial q_1} & \frac{\partial^0 x_P}{\partial q_2} & \frac{\partial^0 x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^0Z_1 & {}^0Z_2 & 0 & {}^0Z_4 & {}^0Z_5 & {}^0Z_6 \end{pmatrix} $$

$$ \begin{bmatrix} -c_1d_2 - s_1s_2d_3 & c_1c_2d_3 & c_1s_2 & 0 & 0 & 0 \\ -s_1d_2 + c_1s_2d_3 & s_1c_2d_3 & s_1s_2 & 0 & 0 & 0 \\ 0 & -s_2d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1s_2 & -c_1c_2s_4 - s_1c_4 & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2c_5 \\ 0 & c_1 & 0 & s_1s_2 & -s_1c_2s_4 + c_1c_4 & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 \\ 1 & 0 & 0 & c_2 & s_2s_4 & -s_2c_4s_5 + c_5c_2 \end{bmatrix} $$

## Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = \begin{pmatrix} J_1 & J_2 & \cdots & J_n \end{pmatrix}$$

$$\det(J) = 0$$

$$\det\left({}^{i}J\right) = \det\left({}^{j}J\right)$$

## Kinematic Singularity

$${}^{B}J = \begin{pmatrix} {}^{B}_{A}R & 0 \\ 0 & {}^{B}_{A}R \end{pmatrix} {}^{A}J$$

$$\det[{}^{B}J] \equiv \det[{}^{A}J]$$

$$\boxed{\det\left({}^{i}J\right) = \det\left({}^{j}J\right)}$$

## Singular Configurations

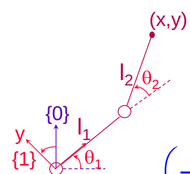
$$\det[J(q)] = 0$$

$\Longrightarrow$ Singular Configurations

$$\det[J(q)] = S_1(q) S_2(q) \ldots S_s(q) = 0$$

$$\Rightarrow \quad \begin{array}{l} S_1(q) = 0 \\ S_2(q) = 0 \\ \vdots \\ S_s(q) = 0 \end{array}$$

## Example (Kinematic Singularities)




$$x = l_1 C1 + l_2 C12$$

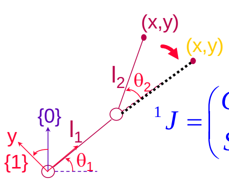
$$y = l_1 S1 + l_2 S12$$

$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

$$\det(J) = l_1 l_2 S2$$

Singularity at $q_2 = k\pi$

## Example (Kinematic Singularities)




$${}^{1}J = {}^{1}_{0}R \; {}^{0}J$$

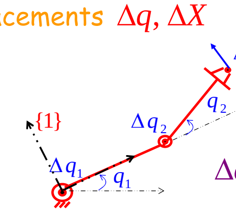
$${}^{1}J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} -l_2 S2 & -l_2 S2 \\ l_1 + l_2 C2 & l_2 C2 \end{pmatrix}$$

At Singularity $\quad {}^{1}J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$

$$\begin{cases} {}^{1}\delta x = 0 \\ {}^{1}\delta y = (l_1 + l_2)\delta\theta_1 + l_2 \delta\theta_2 \end{cases}$$

## Small Displacements $\Delta q$, $\Delta X$

$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

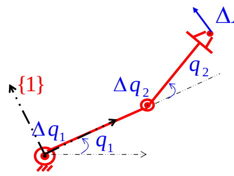



$$\Delta q = J^{-1} \Delta X$$

small $\theta_2$

$$J^{-1}_{(1)} \cong \begin{pmatrix} \frac{1}{l_1 \theta_2} & \frac{1}{l_1} \\ -\frac{l_1 + l_2}{l_1 l_2 \theta_2} & -\frac{1}{l_1} \end{pmatrix}$$

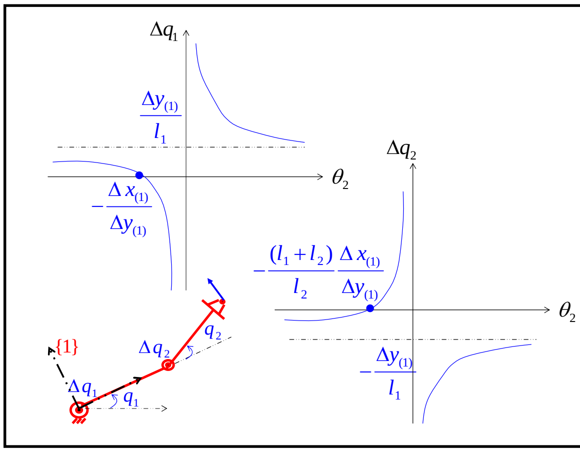
## Small Displacements $\Delta q, \Delta X$



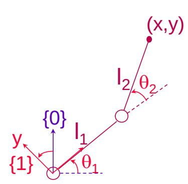

$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\Delta q_1 = \frac{\Delta x_{(1)}}{l_1} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$

$$\Delta q_2 = \frac{(l_1 + l_2)\Delta x_{(1)}}{l_1 l_2} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$



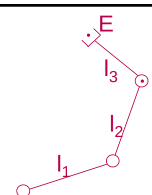

## Kinematic Singularities (reduced matrix)




$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\det(J) = l_1 l_2 S2 \qquad J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

Singularity at $q_2 = k\pi$



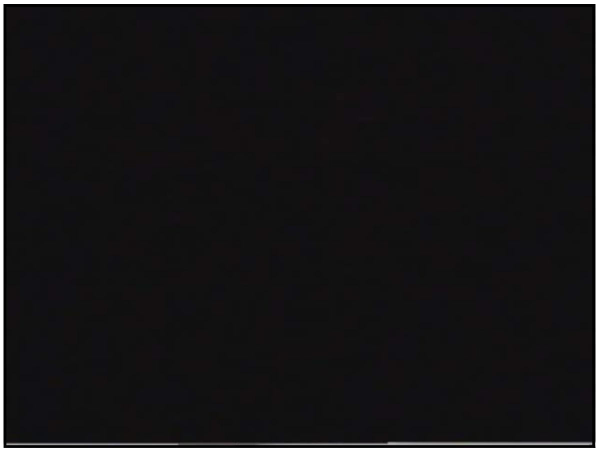

$$^0J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$^0J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

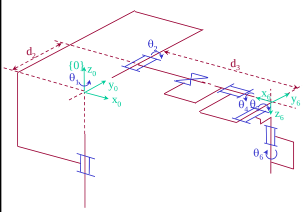
$$^0J_E = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{pmatrix}$$

# Movie Segment

Automatic Parallel Parking, INRIA, ICRA 1999 video proceedings



## Stanford Scheinman Arm




| i | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | -90 | 0 | $d_2$ | $\theta_2$ |
| 3 | 90 | 0 | $d_3$ | 0 |
| 4 | 0 | 0 | 0 | $\theta_4$ |
| 5 | -90 | 0 | 0 | $\theta_5$ |
| 6 | 90 | 0 | 0 | $\theta_6$ |

$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Forward Kinematics: ${}^{0}_{N}T = {}^{0}_{1}T\ {}^{1}_{2}T \ \ldots\ {}^{N-1}_{\ \ N}T$

## Stanford Scheinman Arm Jacobian

$$ {}^{0}J = \begin{pmatrix} \dfrac{\partial\, {}^{0}x_P}{\partial q_1} & \dfrac{\partial\, {}^{0}x_P}{\partial q_2} & \dfrac{\partial\, {}^{0}x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^{0}Z_1 & {}^{0}Z_2 & 0 & {}^{0}Z_4 & {}^{0}Z_5 & {}^{0}Z_6 \end{pmatrix} $$

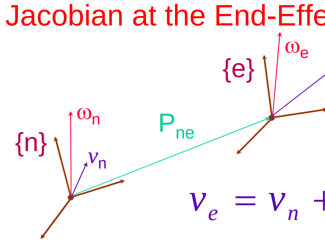
$$ \begin{bmatrix} -c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ -s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & -s_2 c_4 s_5 + c_5 c_2 \end{bmatrix} $$

## Stanford Scheinman Arm Jacobian

$$ \theta_5 = k\pi $$

$$ J = \begin{bmatrix} -c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ -s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 s_2 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 s_2 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & c_2 \end{bmatrix} $$

## Jacobian at the End-Effector




$$ v_e = v_n + \omega_n \times P_{ne} $$

$$ \begin{cases} v_e = v_n - P_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases} $$

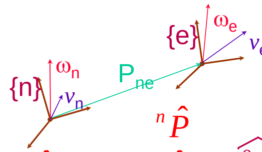
$$\begin{cases} v_e = v_n - P_{ne} \times \omega_n \\ \omega_e = \omega_n \end{cases}$$

$$\begin{pmatrix} v_e \\ \omega_e \end{pmatrix} = \left(\begin{array}{c|c} I & -\hat{P}_{ne} \\ \hline O & I \end{array}\right) \begin{pmatrix} v_n \\ \omega_n \end{pmatrix}$$

$$J_e \dot{q} = \left(\begin{array}{c|c} I & -\hat{P}_{ne} \\ \hline O & I \end{array}\right) J_n \dot{q}$$

$$J_e = \left(\begin{array}{c|c} I & -\hat{P}_{ne} \\ \hline O & I \end{array}\right) J_n$$

## Cross Product Operator (in diff. frames)




$${}^0J_e = \begin{pmatrix} I & -{}^0\hat{P}_{ne} \\ 0 & I \end{pmatrix} {}^0J_n$$

$${}^n\hat{P}$$

$${}^0\hat{P} \neq {}^0_nR \; {}^n\hat{P}; \qquad \widehat{{}^0P} = (\widehat{{}^0_nR.{}^nP}) \neq {}^0_nR.\widehat{{}^nP}$$

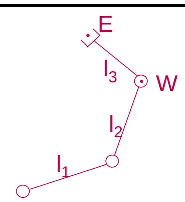
$${}^0P \times {}^0\omega = {}^0_nR.({}^nP \times {}^n\omega)$$

$${}^0\hat{P}.{}^0\omega = {}^0_nR.({}^n\hat{P}.{}^n\omega) = {}^0_nR.({}^n\hat{P}.{}^0_nR^T.{}^0\omega)$$

$$\boxed{{}^0\hat{P} = {}^0_nR \; {}^n\hat{P} \; {}^0_nR^T}$$

$${}^iJ = \begin{pmatrix} {}^i_jR & 0 \\ 0 & {}^i_jR \end{pmatrix} {}^jJ$$

$${}^0J_e = \begin{pmatrix} {}^0_nR & -{}^0_nR \, {}^n\hat{P}_{ne} \, {}^0_nR^T \\ 0 & {}^0_nR \end{pmatrix} {}^nJ_n$$

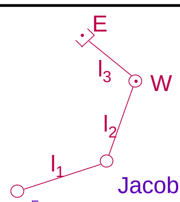



Wrist Point

$$x = l_1c_1 + l_2c_{12}$$
$$y = l_1s_1 + l_2s_{12}$$

End-Effector Point

$$x = l_1c_1 + l_2c_{12} + l_3c_{123}$$
$$y = l_1s_1 + l_2s_{12} + l_3s_{123}$$




Wrist Point

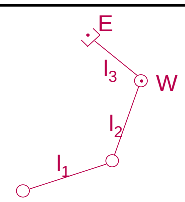
$$x = l_1c_1 + l_2c_{12}$$
$$y = l_1s_1 + l_2s_{12}$$

End-Effector Point

$$x = l_1c_1 + l_2c_{12} + l_3c_{123}$$
$$y = l_1s_1 + l_2s_{12} + l_3s_{123}$$

Jacobian (W)

$$J_W = \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} & 0 \\ l_1c_1 + l_2c_{12} & l_2c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}; \quad {}^0J_E = \begin{pmatrix} I & -{}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0J_W$$



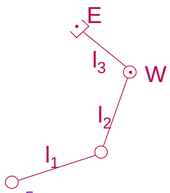

Wrist Point

$$x = l_1c_1 + l_2c_{12}$$
$$y = l_1s_1 + l_2s_{12}$$

End-Effector Point

$$x = l_1c_1 + l_2c_{12} + l_3c_{123}$$
$$y = l_1s_1 + l_2s_{12} + l_3s_{123}$$

$$J_W = \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} & 0 \\ l_1c_1 + l_2c_{12} & l_2c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad {}^0J_E = \begin{bmatrix} -l_1s_1 - l_2s_{12} - l_3s_{123} & -l_2s_{12} - l_3s_{123} & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$




Wrist Point

$$x = l_1 c_1 + l_2 c_{12}$$
$$y = l_1 s_1 + l_2 s_{12}$$

End-Effector Point

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$
$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$J_W = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad {}^0J_E = \begin{pmatrix} I & -{}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0J_W$$

$${}^0P_{WE} = \begin{bmatrix} l_3 c_{123} \\ l_3 s_{123} \\ 0 \end{bmatrix} \Rightarrow {}^0\hat{P}_{WE} = \begin{pmatrix} 0 & 0 & l_3 s_{123} \\ 0 & 0 & -l_3 c_{123} \\ -l_3 s_{123} & l_3 c_{123} & 0 \end{pmatrix}$$

## Resolved Motion Rate Control (Whitney 72)

$$\delta x = J(\theta)\delta\theta$$

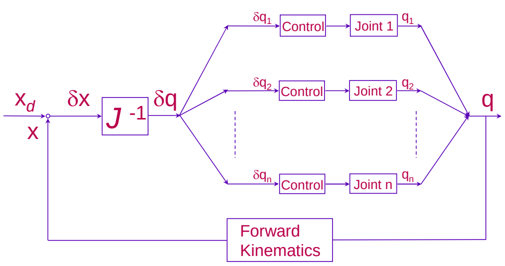
Outside singularities

$$\delta\theta = J^{-1}(\theta)\delta x$$

Arm at Configuration $\theta$

$$x = f(\theta)$$
$$\delta x = x_d - x$$
$$\delta\theta = J^{-1}\delta x$$
$$\theta^{+} = \theta + \delta\theta$$

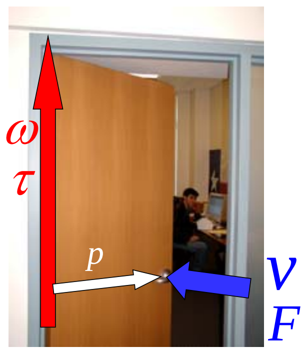
## Resolved Motion Rate Control




## Jacobian

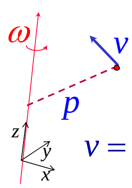
- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

## Angular/Linear – Velocities/Forces

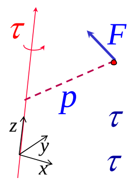



$$v = \omega \times p$$
$$\tau = p \times F$$

## Angular/Linear – Velocities/Forces




$$v = \omega \times p$$
$$v = -\hat{p}\,\omega$$
$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -p_y \\ p_x \end{pmatrix} \dot{\theta}$$
$$v = J\,\dot{\theta}$$

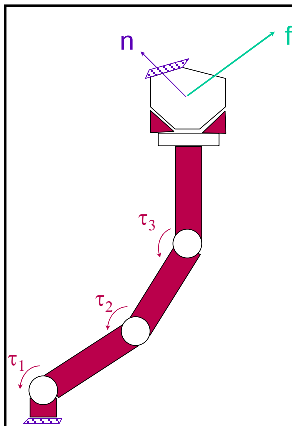


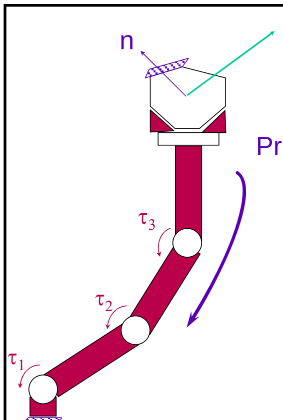

$$\tau = p \times F$$
$$\tau = \hat{p}\ F$$
$$\tau = (-\hat{p})^T\ F$$
$$\tau = \begin{pmatrix} -p_y & p_x \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$
$$\tau = J^T F$$

## Velocity/Force Duality

$$\dot{x} = J\,\dot{\theta}$$

$$\tau = J^T F$$

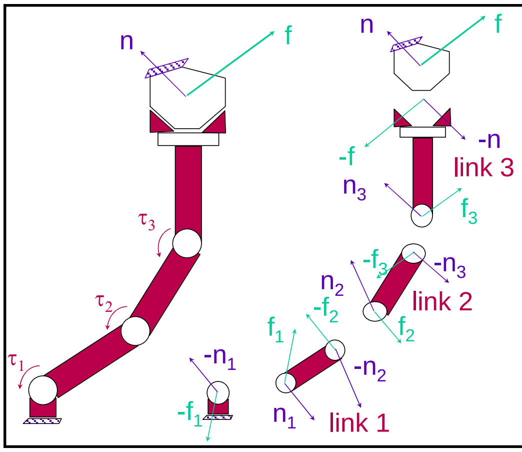


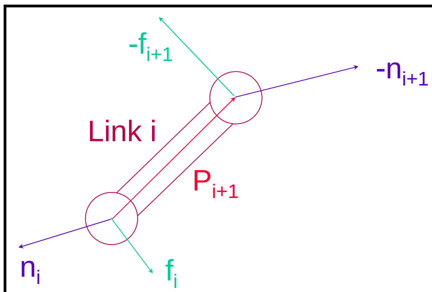



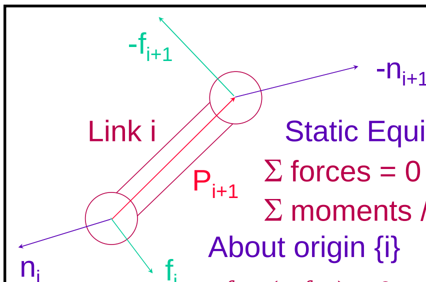

Propagation

Elimination of
Internal forces

Energy Analysis
Virtual Work
Static Equilibrium




Static Equilibrium

Σ forces = 0

Σ moments / a point = 0

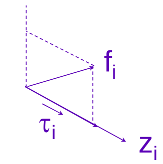
About origin {i}

$$f_i + (-f_{i+1}) = 0$$

$$n_i + (-n_{i+1}) + P_{i+1} \times (-f_{i+1}) = 0$$

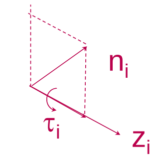
$$f_i = f_{i+1}$$

$$n_i = n_{i+1} + P_{i+1} \times f_{i+1}$$




Prismatic Joint

$$\tau_i = f_i^T Z_i$$



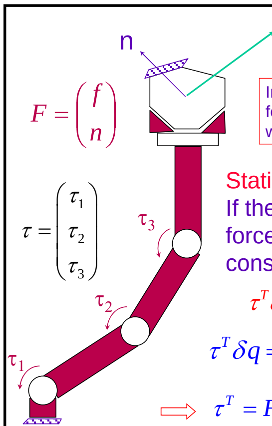

Revolute Joint

$$\tau_i = n_i^T Z_i$$

Algorithm

$$^n f_n = {}^n f$$
$$^n n_n = {}^n n + {}^n P_{n+1} \times {}^n f$$
$$^i f_i = {}_{i+1}^{i}R \, {}^{i+1} f_{i+1}$$
$$^i n_i = {}_{i+1}^{i}R \, {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i$$

## Virtual Work Principal




$$F = \begin{pmatrix} f \\ n \end{pmatrix} \qquad \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$$

Internal forces are workless

$$\delta w = \sum_i f_i \, \delta x_i$$

applied forces — virtual displacements

Static Equilibrium:
If the virtual work done by applied forces is zero in displacements consistent with constraints

$$\tau^T \delta q + (-F)^T \delta x = 0$$

$$\tau^T \delta q = F^T \delta x \quad \text{using} \quad \delta x = J \, \delta q$$

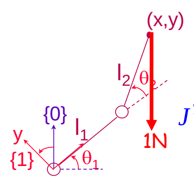
$$\Rightarrow \tau^T = F^T J \Rightarrow \boxed{\tau = J^T F}$$

## Velocity/Force Duality

$$\boxed{\dot{x} = J \, \dot{\theta}}$$

$$\boxed{\tau = J^T F}$$

## Example (Static Forces)




$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

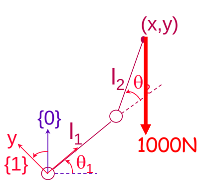
$$J^T = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix}$$

$$\boxed{\tau = J^T F}$$

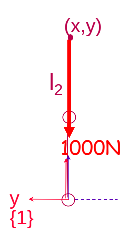
$l_1 = l_2 = 1; \quad \theta_1 = 0; \theta_2 = 60^\circ$

$$\tau = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} l_1 C1 + l_2 C12 \\ l_2 C12 \end{bmatrix} = -\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

## Example (Static Forces)




$$\boxed{\tau = J^T F}$$




$$\tau = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1K \end{bmatrix} = \begin{bmatrix} l_1 C1 + l_2 C12 \\ l_2 C12 \end{bmatrix} (-1K) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$l_1 = l_2 = 1; \quad \theta_1 = 90; \theta_2 = 0^\circ$