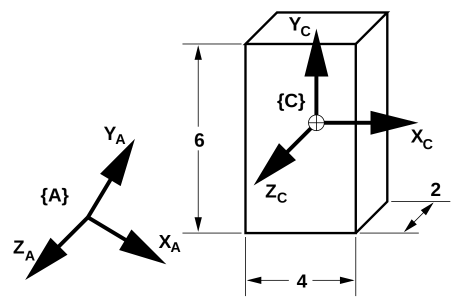
# Introduction to Robotics (CS223A) — Homework #5

(Winter 2007/2008) — Due: **Wednesday, March 05**

1. (a) Derive a formula that transforms an inertia tensor given in some frame $\{C\}$ into a new frame $\{A\}$. The frame $\{A\}$ can differ from frame $\{C\}$ by both translation and rotation. You may assume that frame $\{C\}$ is located at the center of mass.

   (b) Consider, for example, the uniform density box shown below. It has mass $m = 12kg$, and dimensions $6 \times 4 \times 2$:




   Frame $\{C\}$ lies at the center of mass of the box, and the coordinate axes are ligned up with the principal axes of the box. In other words, $\mathbf{Y}_C$ is aligned with the long axis of the box, and $\mathbf{X}_C$ and $\mathbf{Z}_C$ are aligned with the short axes of the box.

   Compute the inertia tensor of the box in frame $\{C\}$.

   *Note: For a frame located at the center of mass and oriented along the principal axes, the inertia tensor for the box of uniform density takes the form:*

$$
{}^{C}I = \begin{bmatrix} \frac{m}{12}(s_y^2 + s_z^2) & 0 & 0 \\ 0 & \frac{m}{12}(s_x^2 + s_z^2) & 0 \\ 0 & 0 & \frac{m}{12}(s_x^2 + s_y^2) \end{bmatrix}
$$

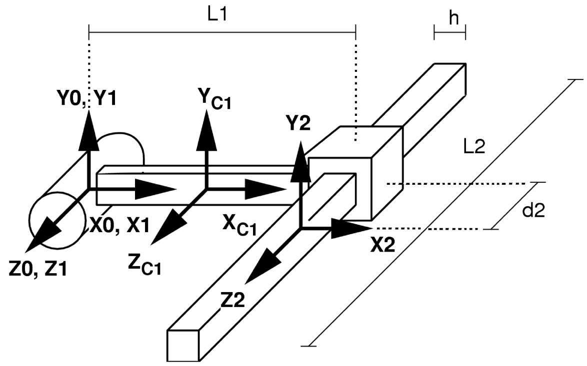
   *where $s_x$, $s_y$ and $s_z$ are the dimensions of the box along the $\mathbf{X}_C$, $\mathbf{Y}_C$ and $\mathbf{Z}_C$ axes, respectively.*

   (c) Given the transformation matrix from $\{C\}$ to $\{A\}$:

$$
{}^{A}_{C}T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

   use your formula from part (a) and your inertia tensor from part (b) to compute the inertia tensor of the box in frame $\{A\}$.

2. In the rest of this problem set, we will walk through the process of finding the equations of motion for a simple manipulator from the Lagrange formulation. Consider the RP spatial manipulator shown below. The links of this manipulator are modeled as bars of uniform density, having square cross-sections of thickness $h$, lengths of $L_1$ and $L_2$, and total masses of $m_1$ and $m_2$, with centers of mass shown. Assume that the joints themselves are massless.




From the derivation in the Lecture Notes, we know that the equations of motion have the form:

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q})\dot{\mathbf{q}}^2 + B(\mathbf{q})\,[\dot{\mathbf{q}}\dot{\mathbf{q}}] + \mathbf{G}(\mathbf{q}) = \tau$$

where $M$ is the mass matrix, $C$ is the matrix of coefficients for centrifugal forces, $B$ is the matrix of coefficients for Coriolis forces, and $\mathbf{G}$ is the vector of gravity forces.

(a) For each link $i$, we have attached a frame $\{C_i\}$ to the center of mass (in this case, frame $\{2\}$ is the same as $\{C_2\}$). Compute kinematics for these frames: that is, calculate the matrices ${}^{0}_{C_1}T$ and ${}^{0}_{C_2}T$.

For a two-link manipulator, the mass matrix has the form

$$M = m_1 J_{v_1}^T J_{v_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega 1}^T\, {}^{C_1}I_1 J_{\omega 1} + J_{\omega 2}^T\, {}^{C_2}I_2 J_{\omega 2}$$

where $J_{v_i}$ is the linear Jacobian of the center of mass of link $i$, $J_{\omega_i}$ is the angular velocity of link $i$, and ${}^{C_i}I_i$ is the inertia tensor of link $i$ expressed in frame $\{C_i\}$.

(b) Calculate ${}^{0}J_{v_1}$ and ${}^{0}J_{v_2}$.

(c) Calculate ${}^{C_1}J_{\omega_1}$ and ${}^{C_2}J_{\omega_2}$.

(d) Calculate ${}^{C_1}I_1$ and ${}^{C_2}I_2$ in terms of the masses and dimensions of the links. You can use the same formula that was given for a box of uniform density in Problem 2(b). Be careful which measurements you use along the axes.

(e) Calculate the mass matrix, $M(\mathbf{q})$. To make your algebra easier, leave the inertia tensors in symbolic form until the end, i.e.

$${}^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}$$

Now we need to calculate the centrifugal and Coriolis forces. We will derive the form directly.

(f) Beginning with the equation in the Lecture Notes,

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_2}\dot{\mathbf{q}} \end{bmatrix},$$

manipulate this equation symbolically into the form

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}]$$

where $C$ and $B$ are matrices in terms of the partial derivatives $m_{ijk}$ of the mass matrix. Don't actually substitute in your answer from part (e) into this equation yet: just leave the elements of these matrices in $m_{ijk}$ symbolic form.

(g) Using your answer to part (e), compute the matrices $C(\mathbf{q})$ and $B(\mathbf{q})$ in terms of the masses, dimensions, and configuration $\mathbf{q}$ of the manipulator.

The last thing that remains is to derive the gravity vector $\mathbf{G}(\mathbf{q})$.

(h) Calculate, ${}^0\mathbf{G}(\mathbf{q})$, the gravity vector in frame $\{0\}$, in terms of the masses, the configuration $\mathbf{q}$, and the gravity constant $g$ ($g$ is positive). Assume that gravity pulls things along the $-\mathbf{Z}_0$ direction. Be careful with your signs.

(i) As a final step, use your answers to parts (e), (h) and (i) to write out the equations of motion as two great big equations:

$$\begin{aligned} \tau_1 &= f_1(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) \\ \tau_2 &= f_2(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) \end{aligned}$$