1. Consider the 1-DOF system described by the equation of motion, \( 4\ddot{x} + 20\dot{x} + 25x = f \).

   (a) Find the natural frequency \( \omega_n \) and the natural damping ratio \( \zeta_n \) of the natural (passive) system \((f = 0)\). What type of system is this (oscillatory, overdamped, etc.)?

   (b) Design a PD controller that achieves critical damping with a closed-loop stiffness \( k_{CL} = 36 \). In other words, let \( f = -k_v\dot{x} - k_p x \), and determine the gains \( k_v \) and \( k_p \). Assume that the desired position is \( x_d = 0 \).

   (c) Assume that the friction model changes from linear \((20\dot{x})\) to Coulomb friction, \(30\text{sign}(\dot{x})\). Design a control system which uses a non-linear model-based portion with trajectory following to critically damp the system at all times and maintain a closed-loop stiffness of \( k_{CL} = 36 \). In other words, let \( f = \alpha f' + \beta \) and \( f' = \ddot{x}_d - k'_v(\dot{x}_d - \dot{x}) - k'_p(x - x_d) \). Then, find \( f, \alpha, \beta, f' \), and \( k'_v \) and \( k'_p \). Note that \( f \) is an \( m \)-mass control, and \( f' \) is a unit-mass control. Use the definition of error, \( e = x - x_d \).

   (d) Given a disturbance force \( f_{dist} = 4 \), what is the steady-state (\( \ddot{e} = \dot{e} = 0 \)) error of the system in part (c)?

2. For a certain RR manipulator, the equations of motion are given by

   \[
   \begin{bmatrix}
   4 + c_2 & 1 + c_2 & \dot{\theta}_1 & \dot{\theta}_2 \\
   1 + c_2 & 1 & \ddot{\theta}_1 & \ddot{\theta}_2
   \end{bmatrix} + \begin{bmatrix}
   -s_2(\ddot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) \\
   s_2\ddot{\theta}_1^2
   \end{bmatrix} = \begin{bmatrix}
   \tau_1 \\
   \tau_2
   \end{bmatrix}
   \]

   (a) Assume that joint 2 is locked at some value \( \theta_2 \) using brakes and joint 1 is controlled with a PD controller, \( \tau_1 = -40\dot{\theta}_1 - 400(\theta_1 - \theta_{1d}) \). What is the minimum and maximum inertia perceived at joint 1 as we vary \( \theta_2 \)? What are the corresponding closed-loop frequencies?

   (b) Still assuming that joint 2 is locked, at what values of \( \theta_2 \) do the minimum and maximum damping ratios occur? What are the minimum and maximum damping ratios?

   (c) Now assume that both joints are free to move, and that this system is controlled by a partitioned PD controller, \( \tau = \alpha\tau' + \beta \). Design a partitioned, trajectory-following controller (one that tracks a desired position, velocity and acceleration) which will provide a closed-loop frequency of 10 rad/sec on joint 1 and 20 rad/sec on joint 2 and be critically damped over the entire workspace. That is, let

   \[
   \tau' = \ddot{\theta}_d - \begin{bmatrix}
   k'_{v1} & 0 \\
   0 & k'_{v2}
   \end{bmatrix} (\dot{\theta} - \dot{\theta}_d) - \begin{bmatrix}
   k'_{p1} & 0 \\
   0 & k'_{p2}
   \end{bmatrix} (\theta - \theta_d)
   \]

   then find the matrices \( \alpha \) and \( \beta \) and the vector \( \tau \), along with the necessary gains \( k'_{v1} \) and \( k'_{p1} \).

   (d) If \( \theta_2 = 180^\circ \), what is the steady-state error vector for a given disturbance torque, \( \tau_{dist} = [2 4]^T \)?