(Winter 2007/2008)

1. **A frame $\{B\}$ and a frame $\{A\}$ are initially coincident. Frame $\{B\}$ is rotated about $\hat{Y}_B$ by an angle θ, and then rotated about the new $\hat{Z}_B$ by an angle ϕ. Determine the 3×3 rotation matrix, ${}^A_BR$, which will transform the coordinates of a position vector from ${}^B\mathbf{P}$, its value in frame $\{B\}$, into ${}^A\mathbf{P}$, its value in frame $\{A\}$.**

   Consider the intermediate frame $\{M\}$ which results after the first rotation:

$$ {}^A_BR = {}^A_MR\,{}^M_BR $$

   Now, the frame transformations from $\{A\}$ to $\{M\}$, and $\{M\}$ to $\{B\}$, are precisely those rotations listed in the question, so we know that ${}^A_MR = R_y(\theta)$ and ${}^M_BR = R_z(\phi)$. Thus:

$$ {}^A_BR = R_y(\theta)R_z(\phi) $$

   Indeed, this is the Y-Z Euler-angle representation for frame $\{B\}$ w.r.t. frame $\{A\}$.

   Written out:

$$ \begin{aligned} {}^A_BR &= \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c\theta c\phi & -c\theta s\phi & s\theta \\ s\phi & c\phi & 0 \\ -s\theta c\phi & s\theta s\phi & c\theta \end{bmatrix} \end{aligned} $$

2. **We are given a single frame $\{A\}$ and a position vector ${}^A\mathbf{P}$ described in this frame. We then transform ${}^A\mathbf{P}$ by first rotating it about $\hat{Z}_A$ by an angle ϕ, then rotating about $\hat{Y}_A$ by an angle θ. Determine the 3×3 rotation matrix operator, $R(\phi, \theta)$, which describes this transformation.**

   Suppose the first rotation converts ${}^A\mathbf{P} \rightarrow {}^A\mathbf{P}'$, and the second rotation converts ${}^A\mathbf{P}' \rightarrow {}^A\mathbf{P}''$. Then we have:

$$ \begin{aligned} {}^A\mathbf{P}' &= R_z(\phi)\,{}^A\mathbf{P} \\ {}^A\mathbf{P}'' &= R_y(\theta)\,{}^A\mathbf{P}' \\ \Rightarrow {}^A\mathbf{P}'' &= R_y(\theta)R_z(\phi)\,{}^A\mathbf{P} \end{aligned} $$

   which gives the result:

$$ R(\phi, \theta) = R_y(\theta)R_z(\phi) $$

   This is the same matrix as question 1. After all, this displacement is mathematically equivalent to a frame transformation using Z-Y Fixed-Angle representation, which in turn is mathematically equivalent to a Y-Z Euler-Angle representation.

3. (a) **Given a transformation matrix:**

$$ {}^B_A T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \cos(\theta) & -\sin(\theta) & 2 \\ 0 & \sin(\theta) & \cos(\theta) & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Find ${}^A_B T$

Use Equation 1.26 from page 20 of the Lecture Notes, to get:

$$ {}^B_A T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & c\theta & s\theta & -2c\theta - 3s\theta \\ 0 & -s\theta & c\theta & 2s\theta - 3c\theta \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(b) **Given** $\theta = 45°$ **and** ${}^B P = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}^T$, **compute** ${}^A P$**.**

${}^A P = \begin{bmatrix} 3 & 4.24 & 0 \end{bmatrix}^T$

4. **Given the following** 3×3 **matrix:**

$$ R = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} $$

(a) **Show that it is a rotation matrix.**

This can be shown by $R^T R = I$, where I is the identity matrix.

$$ R^T R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

(b) **Determine a unit vector that defines the axis of rotation and the angle (in degrees) of rotation.**

Apply equations 1.52, 1.53 on page 34 of the Lecture Notes, to get: $angle = 62.8^o$, and $axis = \begin{bmatrix} -0.679 & 0.679 & -0.281 \end{bmatrix}^T$.

(c) **What are the Euler parameters** $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ **of** R**?**

Apply equations 1.54 – 1.57 on pages 34-35 of the Lecture Notes: $\varepsilon_1 = -0.354$, $\varepsilon_2 = 0.354$, $\varepsilon_3 = -0.146$, $\varepsilon_4 = 0.854$.