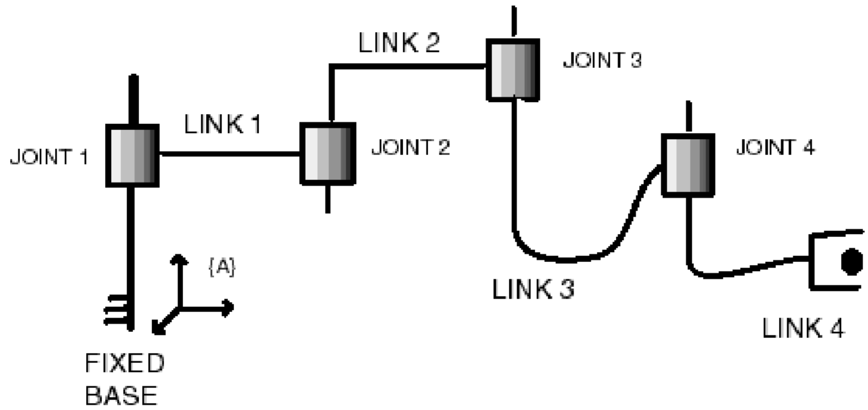
Introduction to Robotics (CS223A) Homework #2 Solution

(Winter 2007/2008)

1. **The following sketch represents a generic open, serial, kinematic-chain.**



Here each kinematic joint connects two adjacent members. Assume that the relative displacement between adjacent members $i-1$ and i is described by an operator T_i that is a 4x4 matrix whose elements are computed in a coordinate frame $\{A\}$ fixed to the base of the chain. Now, if each member is displaced in sequence, *starting from the free end*, the displacement operator for the resultant total displacement of the free end will be given by $T_1T_2T_3T_4$. (Note: In this problem you are to use only displacements operators, not coordinate transformations)

However, if the displacements are done in the reverse order, ie. *starting at the fixed end*, and moving in the sequence 1, 2, 3, 4, then the operators T_2, T_3 , and T_4 no longer represent the actual displacements.

Determine, in terms of the original T_i:

(a) **The operator for joint 2, when its displacement is done *after* the displacement in joint 1. Let us call this operator T_2'**

The first thing to recognize is that this entire question is based on displacement operators, *not* frame transformations – this concept was discussed Lecture Notes section 1.2.5, and is vital to solving this question.

Now, when joint 1 moves it causes a displacement (given by T_1) which affects the entire manipulator – including the rotation axes of joints 2, 3, and 4. To find the new rotation axis corresponding to joint 2, we use a similarity transform:

$$T_2' = T_1T_2T_1^{-1}$$

(b) **The operator for joint 3 when its displacement follows the displacement in joints 1 and 2 (from part (a)). Let us call this operator T_3'**

Joint 3 gets displaced first by joint 1 (T_1), followed by joint 2 from part (a) (T_2'). So when we do a similarity transform on T_3, we use the combined displacement matrix $T_2'T_1$. This means:

$$T_3' = (T_2'T_1)T_3(T_2'T_1)^{-1}$$

Using the answer from part (a) to substitute in for T_2' gives:

$$\begin{aligned} T_3' &= (T_1T_2)T_3(T_1T_2)^{-1} \\ &= T_1T_2T_3T_2^{-1}T_1^{-1} \end{aligned}$$

(c) **The operator for joint 4 when its displacement follows the displacement in joints 1, 2 and 3 (from part (b)). Let us call this operator T_4'**

The reasoning for this part is the same as before. Joint 4 gets displaced first by T_1, followed by T_2', and T_3'. So when we do the similarity transform, we need to use $T_3'T_2'T_1$. This gives:

$$T_4' = (T_3'T_2'T_1)T_4(T_3'T_2'T_1)^{-1}$$

Plug in the results for T_2' and T_3' from parts (a) and (b) respectively, and then simplify:

$$\begin{aligned} T_4' &= (T_1T_2T_3T_2^{-1}T_1^{-1}T_1T_2T_1^{-1}T_1)T_4(T_1T_2T_3T_2^{-1}T_1^{-1}T_1T_2T_1^{-1}T_1)^{-1} \\ &= (T_1T_2T_3)T_4(T_1T_2T_3)^{-1} \\ &= T_1T_2T_3T_4T_3^{-1}T_2^{-1}T_1^{-1} \end{aligned}$$

(d) **Using your results for parts (a), (b) and (c), show that the resulting displacement operator for the free end is still $T_1T_2T_3T_4$**

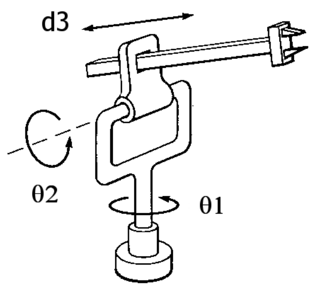
First joint 1 moved, then joint 2 from part (a), then joint 3 from part (b), then joint 4 from part (c). The final displacement of the end effector is given by:

$$T_{final} = T_4'T_3'T_2'T_1$$

Plug in the expressions from the previous parts and simplify:

$$\begin{aligned} T_{final} &= (T_1T_2T_3T_4T_3^{-1}T_2^{-1}T_1^{-1})(T_1T_2T_3T_2^{-1}T_1^{-1})(T_1T_2T_1^{-1})T_1 \\ &= T_1T_2T_3T_4 \end{aligned}$$

2. **Consider the following RRP manipulator (figure courtesy of J. J. Craig):**

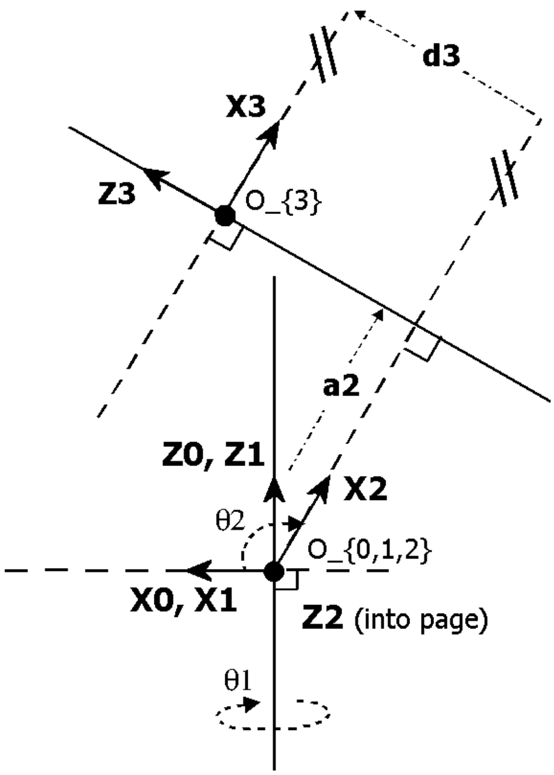


(a) **Draw a schematic of this manipulator, with the axes of frames $\{0\}$ through $\{3\}$ labeled. Also, include the parameters θ_1, θ_2, a_2, and d_3 on your schematic.**

Assume that in this diagram, the slider bar is parallel to the ground and that this is the configuration where $\theta_1 = 0$, $\theta_2 = 90^o$.

The following diagram is one possible schematic. The base frame {0} was chosen to overlap with frame {1} when $\theta_1 = 0$. In the final frame {3}, the x-axis was chosen to make $\theta_3 = 0$. It is acceptable (though not preferable for this class) to do otherwise with the start and end frames.

Indeed, there are other ways in which schematics may vary. To begin with, it can be drawn for a different set of joint values from those depicted below. Schematics may vary based on the chosen direction ("up" or "down") for the z-axes and x-axes. Also, since joint 3 is prismatic, we could use a different axis for this joint as long as it is parallel to the one depicted. In this case, a_2 may be different for each person (some people may even put it as zero).



(b) **Write down the Denavit-Hartenberg parameters for this manipulator, in the form of a table:**

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	90^o	0	θ_2
3	a_2	90^o	d_3	0

(c) **Derive the forward kinematics for this manipulator — that is, find ${}^0_3T$.**

Start with the transformations between intermediate frames:

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

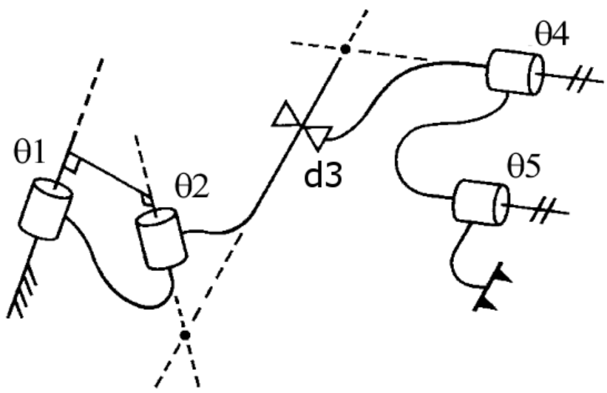
$$
{}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Multiply them out to get ${}^{0}_{3}T$:

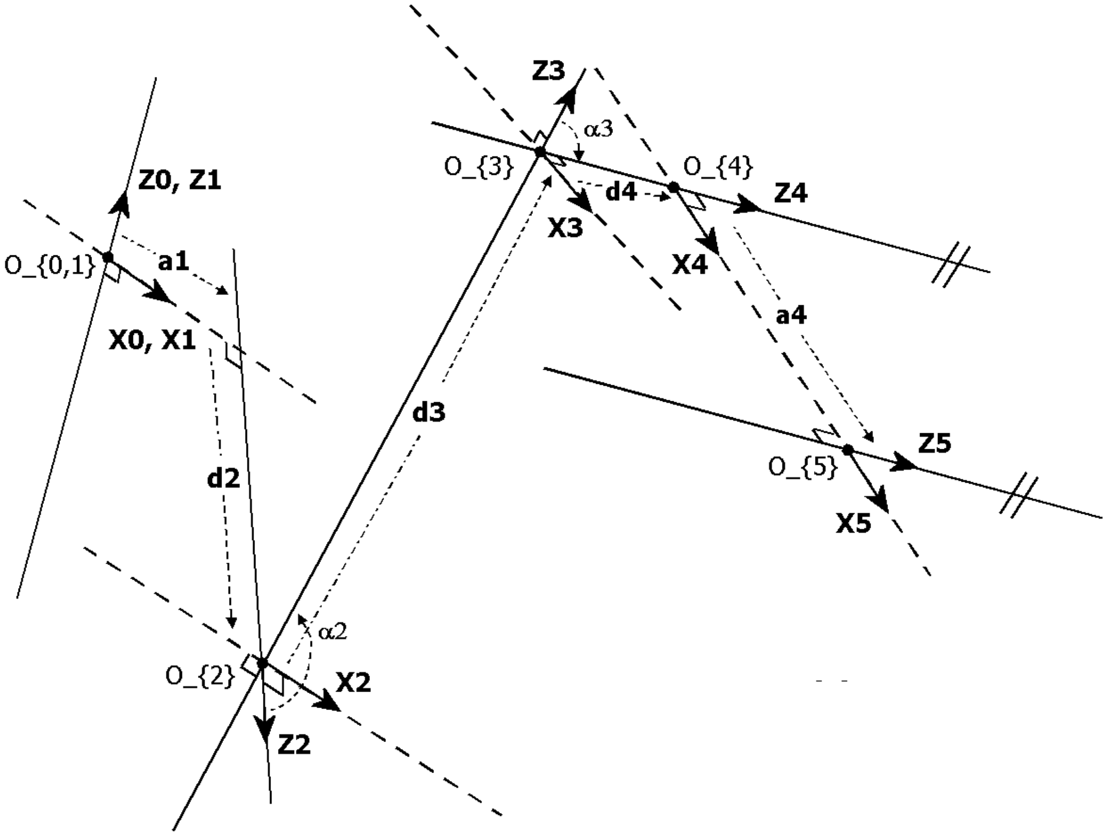
$$
\begin{aligned}
{}^{0}_{3}T &= {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \\
&= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_1c_2 & s_1 & c_1s_2 & c_1c_2a_2 + c_1s_2d_3 \\ s_1c_2 & -c_1 & s_1s_2 & s_1c_2a_2 + s_1s_2d_3 \\ s_2 & 0 & -c_2 & s_2a_2 - c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

3. **Consider the following 2RP2R manipulator (figure courtesy of J. J. Craig):**



(a) **Draw a schematic of this manipulator, with the axes of frames $\{0\}$ through $\{5\}$ labeled. Include all non-zero Denavit-Hartenberg parameters and the joint variables. Draw your schematic in the position where, as far as possible, the angles θ_i are in their zero positions.**

One possible schematic is shown below. In this particular case, since axes 4 and 5 are parallel, their common perpendicular can be drawn anywhere, and therefore d_4 may differ between people (it may even be zero). Another peculiarity is the slider axis (axis 3), which may be relocated anywhere, as long as it is parallel to the one depicted below. Other variations will include choices of up/down for the coordinate axes and placement of base and end frames.



(b) **Write down the Denavit-Hartenberg parameters for this manipulator, in the form of a table:**

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	a_1	α_1	d_2	θ_2
3	0	α_2	d_3	θ_3
4	0	α_3	d_4	θ_4
5	a_4	0	0	θ_5

Note: besides the joint variables (θ_1, θ_2, d_3, θ_4, θ_5) all the other parameters above are constants.