(Winter 2007/2008)

1. **You are given that a certain RPR manipulator has the following transformation matrices, where $\{E\}$ is the frame of the end effector.**

$$
{}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^0_3T = \begin{bmatrix} c_1c_3 & -c_1s_3 & -s_1 & L_1c_1 - s_1d_2 \\ s_1c_3 & -s_1s_3 & c_1 & L_1s_1 + c_1d_2 \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

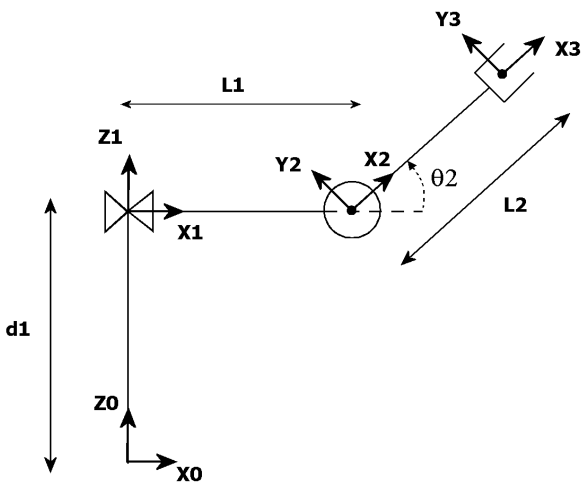
$$
{}^0_ET = \begin{bmatrix} -s_1 & c_1s_3 & c_1c_3 & L_1c_1 + L_2c_1c_3 - s_1d_2 \\ c_1 & s_1s_3 & s_1c_3 & L_1s_1 + L_2s_1c_3 + c_1d_2 \\ 0 & c_3 & -s_3 & -L_2s_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Derive the basic Jacobian relating joint velocities to the end-effector's linear and angular velocities in frame $\{0\}$.

For J_v, we simply differentiate the position of the end-effector expressed in frame $\{0\}$, which is the last column of ${}^0_4T$. For J_ω we take the z-vectors from ${}^0_1T$ and ${}^0_3T$, and since joint 2 is prismatic, it doesn't contribute.

$$
J_v = \begin{bmatrix} -L_1s_1 - L_2s_1c_3 - c_1d_2 & -s_1 & -L_2c_1s_3 \\ L_1c_1 + L_2c_1c_3 - s_1d_2 & c_1 & -L_2s_1s_3 \\ 0 & 0 & -L_2c_3 \end{bmatrix} \qquad J_\omega = \begin{bmatrix} 0 & 0 & -s_1 \\ 0 & 0 & c_1 \\ 1 & 0 & 0 \end{bmatrix}
$$

2. **Consider the planar PR manipulator shown here:**



   (a) **Find the origin of frame $\{3\}$ expressed in terms of frame $\{0\}$, that is ${}^0\mathbf{P}_{3org}$.** ***Tip: you can derive this geometrically, if you want to avoid going through DH parameters.***

It's simplest to do this geometrically. Note: I will often refer to the origin of frame {3} as the "tip".

In the base frame {0}, the x-coordinate of the tip is found by projecting the link L_2 onto the $\hat{X}_0$ axis and adding to L_1. Similarly, the z-coordinate is found by projecting L_2 onto the $\hat{Z}_0$ axis and adding to the slider displacement d_1. Thus, in frame {0}:

$$^0\mathbf{P}_{3org} = \begin{bmatrix} x_{tip} \\ z_{tip} \end{bmatrix} = \begin{bmatrix} L_1 + L_2\cos(\theta_2) \\ d_1 + L_2\sin(\theta_2) \end{bmatrix}$$

(b) **Give the 2×2 Jacobian that relates the joint velocities to the linear velocity of $^0\mathbf{P}_{3org}$.**

$$J = \begin{bmatrix} \frac{\partial x_{tip}}{\partial d_1} & \frac{\partial x_{tip}}{\partial \theta_2} \\ \frac{\partial z_{tip}}{\partial d_1} & \frac{\partial z_{tip}}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} 0 & -L_2\sin(\theta_2) \\ 1 & L_2\cos(\theta_2) \end{bmatrix}$$

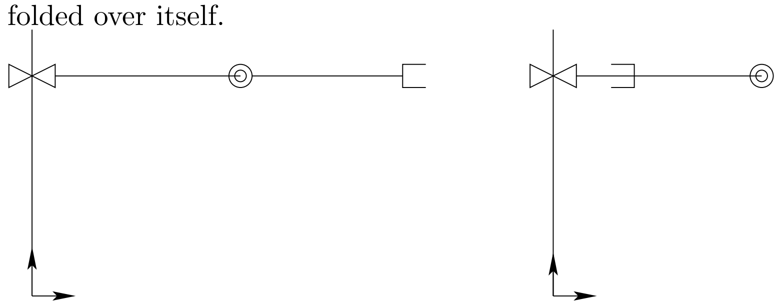
(c) **For what joint values is the manipulator at a singularity? What motion is restricted at this singularity?**
The singularity occurs at a configuration when $det(J) = 0$.

$$\begin{aligned} det(J) &= J_{11}J_{22} - J_{12}J_{21} \\ &= L_2\sin(\theta_2) \end{aligned}$$

So the singularity occurs whenever $\sin(\theta_2) = 0$, which corresponds to:

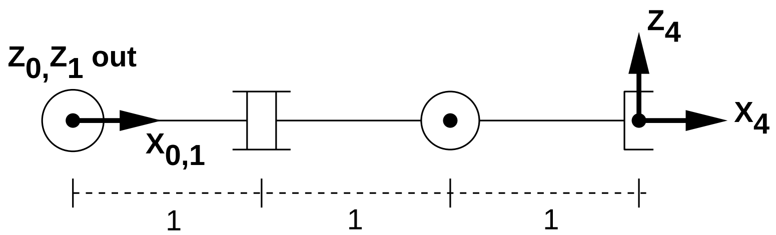
$$\theta_2 = 0^o \text{ or } \pm 180^o$$

These two situations are portrayed below; either the arm is extended out completely or folded over itself.



In both cases, **the end-effector cannot move instantaneously in the $\hat{X}_0$ direction**, ie. the joints cannot produce a velocity component in the $\hat{X}_0$ direction.

3. **Consider the RRR manipulator shown here:**



Note: in the figure, the numbers below the links represent the lengths.

(a) **Find the DH parameters for this manipulator. Remember to assign the interior frames of this manipulator using the conventions discussed in class.**

Note: In preparation for part (b), one may add an extra row to the bottom of the table corresponding to frame {4}. Since the question didn't specifically ask for this extra row, it's not necessary to have it. After all, frame {4} is fixed wrt to frame {3} and the transformation between the two can be found (if one so wishes) by inspection.

i	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	θ_1	0
2	-90^o	1	θ_2	0
3	90^o	1	θ_3	0
4	-90^o	1	0	0

(b) **Derive the forward kinematics, ${}^0_4T$, of this manipulator.**

Use the above table to compute the DH transformation matrices. In preparation for computing the Jacobian in part (c), one may also compute the ${}^0_iT$ for each frame $\{i\}$.

$$
\begin{aligned}
{}^0_1T &= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^1_2T &= \begin{bmatrix} c_2 & -s_2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\Rightarrow {}^0_2T &= {}^0_1T\,{}^1_2T = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 & c_1 \\ s_1c_2 & -s_1s_2 & c_1 & s_1 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^2_3T &= \begin{bmatrix} c_3 & -s_3 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\Rightarrow {}^0_3T &= {}^0_2T\,{}^2_3T = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1c_2s_3 - s_1c_3 & c_1s_2 & c_1c_2 + c_1 \\ s_1c_2c_3 + c_1s_3 & -s_1c_2s_3 + c_1c_3 & s_1s_2 & s_1c_2 + s_1 \\ -s_2c_3 & s_2s_3 & c_2 & -s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^3_4T &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\Rightarrow {}^0_4T &= {}^0_3T\,{}^3_4T = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1s_2 & -c_1c_2s_3 - s_1c_3 & c_1c_2c_3 - s_1s_3 + c_1c_2 + c_1 \\ s_1c_2c_3 + c_1s_3 & -s_1s_2 & -s_1c_2s_3 + c_1c_3 & s_1c_2c_3 + c_1s_3 + s_1c_2 + s_1 \\ -s_2c_3 & -c_2 & s_2s_3 & -s_2c_3 - s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

(c) **Find the basic Jacobian, J_0, for this manipulator.**

Use the explicit form: take derivatives of the end-effector position (the last column of ${}^0_4T$) for J_v, and use the Z-axes of each frame (3rd column of each ${}^0_iT$ matrix calculated above) for J_w.

$$J_0 = \begin{bmatrix} \frac{\partial {}^0\mathbf{P}_{4org}}{\partial\theta_1} & \frac{\partial {}^0\mathbf{P}_{4org}}{\partial\theta_2} & \frac{\partial {}^0\mathbf{P}_{4org}}{\partial\theta_3} \\ {}^0\mathbf{Z}_1 & {}^0\mathbf{Z}_2 & {}^0\mathbf{Z}_3 \end{bmatrix}$$

$$= \begin{bmatrix} -s_1c_2c_3 - c_1s_3 - s_1c_2 - s_1 & -c_1s_2c_3 - c_1s_2 & -c_1c_2s_3 - s_1c_3 \\ c_1c_2c_3 - s_1s_3 + c_1c_2 + c_1 & -s_1s_2c_3 - s_1s_2 & -s_1c_2s_3 + c_1c_3 \\ 0 & -c_2c_3 - c_2 & s_2s_3 \\ 0 & -s_1 & c_1s_2 \\ 0 & c_1 & s_1s_2 \\ 1 & 0 & c_2 \end{bmatrix}$$

(d) **Find ${}^1J_v$, the position Jacobian matrix expressed in frame {1}.**
Compute ${}^1J_v$ by rotating ${}^0J_v$ (the top half of J_0) from frame {0} to frame {1}. Note that this rotation matrix is the inverse (or transpose, in this case) of what appears in ${}^0_1T$ (computed in part (b)), ie. we need to use ${}^1_0R$, not ${}^0_1R$.

$$\begin{aligned} {}^1J_v &= {}^1_0R\,{}^0J_v \\ &= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_1c_2c_3 - c_1s_3 - s_1c_2 - s_1 & -c_1s_2c_3 - c_1s_2 & -c_1c_2s_3 - s_1c_3 \\ c_1c_2c_3 - s_1s_3 + c_1c_2 + c_1 & -s_1s_2c_3 - s_1s_2 & -s_1c_2s_3 + c_1c_3 \\ 0 & -c_2c_3 - c_2 & s_2s_3 \end{bmatrix} \\ &= \begin{bmatrix} -s_3 & -s_2c_3 - s_2 & -c_2s_3 \\ c_2c_3 + c_2 + 1 & 0 & c_3 \\ 0 & -c_2c_3 - c_2 & s_2s_3 \end{bmatrix} \end{aligned}$$

(e) **Use the matrix that you found in part (d) to find the singularities (with respect to linear velocity) of this manipulator.**
The singularities occur when the determinant of the Jacobian (in *any* frame) is zero. Since J_v is simpler in frame {1}, look at the determinant of ${}^1J_v$:

$$\begin{aligned} det({}^1J_v) &= \begin{vmatrix} -s_3 & -s_2c_3 - s_2 & -c_2s_3 \\ c_2c_3 + c_2 + 1 & 0 & c_3 \\ 0 & -c_2c_3 - c_2 & s_2s_3 \end{vmatrix} \\ &= (-s_3)(c_2c_3^2 + c_2c_3) - (c_2c_3 + c_2 + 1)(-s_2^2s_3c_3 - s_2^2s_3 - c_2^2s_3c_3 - c_2^2s_3) \\ &= s_3(c_2c_3 + c_2 + c_3 + 1) \\ &= s_3(1 + c_3)(1 + c_2) \end{aligned}$$

Setting the first term in the above expression to zero gives:

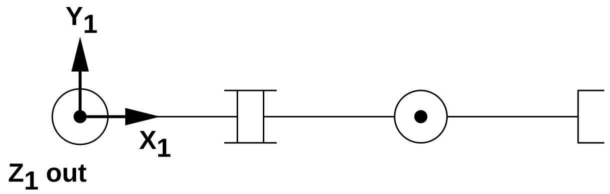
$$\sin(\theta_3) = 0 \;\Rightarrow\; \theta_3 = 0^o \text{ or } \pm 180^o$$

Setting the second term to zero gives the same result as above.
Setting the third term to zero gives:

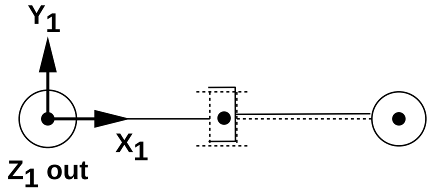
$$\cos(\theta_2) = -1 \;\Rightarrow\; \theta_2 = \pm 180^o$$

(f) **For each type of singularity that you found in part (e), explain the physical interpretation of the singularity, by sketching the arm in a singular configuration and describing the resulting limitation on its movement.**

The singularity $\theta_3 = 0$ is when the outer half of the arm is outstretched. In this position there is no motion possible in the $\hat{X}_3$ axis.



The singularity $\theta_3 = \pm 180^o$ is when the outer link of the arm is folded in on itself. There are two restrictions in this position. First, just as in the outstretched case, there is no motion possible in the $\hat{X}_3$ direction. Second, because the last two links have the same length, the end-effector is overlapping with joint 2. As a result there is no motion possible in the $\hat{Z}_1$ direction.



The singularity $\theta_2 = \pm 180^o$ is when half the arm is folded in over itself, causing joints 1 and 3 to overlap. In this position, joints 1 and 3 have the same effect on the end-effector – as if joint 3 doesn't exist. The ensuing loss of motion can be see in frame $\{3\}$: there is no motion possible in the $\hat{X}_3$ direction.