EE364a Review Session 1

administrative info:

• office hours: tue 4-6pm, wed 4-8pm, packard 277

• review session: example problems and hw hints

• homeworks due thursdays by 5pm

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Combinations and hulls

\[ y = \theta_1 x_1 + \cdots + \theta_k x_k \] is a

- linear combination of \( x_1, \ldots, x_k \)
- affine combination if \( \sum_i \theta_i = 1 \)
- convex combination if \( \sum_i \theta_i = 1, \theta_i \geq 0 \)
- conic combination if \( \theta_i \geq 0 \)

(linear, affine, \ldots) hull of \( S = \{x_1, \ldots, x_k\} \) is a set of all
(linear, affine, \ldots) combinations from \( S \)

- linear hull: \( \text{span}(S) \)
- affine hull: \( \text{aff}(S) \)
- convex hull: \( \text{conv}(S) \)
- conic hull: \( \text{cone}(S) \)
example: a few simple relations:

\[ \text{conv}(S) \subseteq \text{aff}(S) \subseteq \text{span}(S), \quad \text{conv}(S) \subseteq \text{cone}(S) \subseteq \text{span}(S). \]

example: \( S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subseteq \mathbb{R}^3 \)

what is the linear hull? affine hull? convex hull? conic hull?

• linear hull: \( \mathbb{R}^3 \).

• affine hull: hyperplane passing through \((1, 0, 0), (0, 1, 0), (0, 0, 1)\).

• convex hull: triangle with vertices at \((1, 0, 0), (0, 1, 0), (0, 0, 1)\).

• conic hull: \( \mathbb{R}_+^3 \)
Important rules

- **intersection**

  \[ S_\alpha \text{ is } \begin{pmatrix} \text{subspace} \\ \text{affine} \\ \text{convex} \\ \text{convex cone} \end{pmatrix} \quad \text{for } \alpha \in \mathcal{A} \implies \bigcap_{\alpha \in \mathcal{A}} S_\alpha \text{ is } \begin{pmatrix} \text{subspace} \\ \text{affine} \\ \text{convex} \\ \text{convex cone} \end{pmatrix} \]

  **example:** a *polyhedron* is intersection of a finite number of halfspaces and hyperplanes.

- **functions that preserve convexity**

  **examples:** affine, perspective, and linear fractional functions. If \( C \) is convex, and \( f \) is an affine/perspective/linear fractional function, then \( f(C) \) is convex and \( f^{-1}(C) \) is convex.
Quantized measurements

consider the measurement setup,

\[ y = 0.1 \text{floor}(10Ax) \]

where \( x \in \mathbb{R}^2 \) is the input, \( y \in \mathbb{R}^5 \) are the measurements, and \( A \in \mathbb{R}^{5 \times 2} \).

- given a measurement \( y \), we want to find the set of inputs that are consistent with the measurements. i.e., the set

\[ \mathcal{X} = \{ x \mid 0 \leq a_i^T x - y_i \leq 0.1, i = 1, \ldots, 5 \}. \]

we can explore this set by simulating, and plotting points that are inside the set. we randomly choose an \( x \in \mathbb{R}^2 \). if \( x \) is consistent with \( y \), then we plot \( x \). we repeat this a number of times. in the following plot, the blue circles represent points inside \( \mathcal{X} \), and the red dot is the least squares solution, \( x_{ls} = A^\dagger y \).
Quantized measurements

\[ x_1 \]

\[ x_2 \]
from the simulations we suspect that $\mathcal{X}$ is a polyhedron. *i.e. *,

$$
\mathcal{X} = \{ x \mid Fx \leq g \}.
$$

it is easy to show that,

$$
F = \begin{bmatrix}
-a_1^T \\
a_1^T \\
\vdots \\
-a_5^T \\
a_5^T
\end{bmatrix}, \quad g = \begin{bmatrix}
-y_1 \\
y_1 + 0.1 \\
\vdots \\
-y_5 \\
y_5 + 0.1
\end{bmatrix}.
$$
Solution set of a quadratic inequality

let \( C \subseteq \mathbb{R}^n \) be the solution set of a quadratic inequality,

\[
C = \{ x \in \mathbb{R}^n \mid x^T Ax + b^T x + c \leq 0 \},
\]

with \( A \in \mathbb{S}^n \), \( b \in \mathbb{R}^n \), and \( c \in \mathbb{R} \).

• show that \( C \) is convex if \( A \succeq 0 \).

we will show that the intersection of \( C \) with an arbitrary line \( \{ \hat{x} + tv \mid t \in \mathbb{R} \} \) is convex. we have,

\[
(\hat{x} + tv)^T A(\hat{x} + tv) + b^T (\hat{x} + tv) + c = \alpha t^2 + \beta t + \gamma
\]

where,

\[
\alpha = v^T Av, \quad \beta = b^T v + 2\hat{x}^T Av, \quad \gamma = c + b^T \hat{x} + \hat{x}^T A\hat{x}.
\]
the intersection of $C$ with the line defined by $\hat{x}$ and $v$ is the set

$$\{\hat{x} + tv \mid \alpha t^2 + \beta t + \gamma \leq 0\},$$

which is convex if $\alpha \geq 0$. This is true for any $v$ if $A \succeq 0$. 
Voronoi sets and polyhedral decomposition

let $x_0, \ldots, x_K \in \mathbb{R}^n$. consider the set of points that are closer (in Euclidean norm) to $x_0$ than the other $x_i$, i.e.,

$$V = \{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2, i = 1, \ldots, K\}.$$ 

• what kind of set is $V$?

**answer.** $V$ is a polyhedron. we can express $V$ as $V = \{x \mid Ax \leq b\}$ with

$$A = 2 \begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ \vdots \\ x_K - x_0 \end{bmatrix}, \quad b = \begin{bmatrix} x^T_1 x_1 - x^T_0 x_0 \\ x^T_2 x_2 - x^T_0 x_0 \\ \vdots \\ x^T_K x_K - x^T_0 x_0 \end{bmatrix}.$$ 

(check this!)
Conic hull of outer products

consider the set of rank-\(k\) outer products, defined as

\[
\{XX^T \mid X \in \mathbb{R}^{n \times k}, \ rank X = k\}.
\]

describe its conic hull in simple terms.

**solution.** we have \(XX^T \succeq 0\) and \(\text{rank}(XX^T) = k\). a positive combination of such matrices can have rank up to \(n\), but never less than \(k\). indeed, let \(A\) and \(B\) be positive semidefinite matrices of rank \(k\). suppose \(v \in \mathcal{N}(A + B)\), then

\[
(A + B)v = 0 \iff v^T(A + B)v = 0 \iff v^TAv + v^TBv = 0.
\]

this implies,

\[
v^TAv = 0 \iff Av = 0, \quad v^TBv = 0 \iff Bv = 0.
\]

hence any vector in the \(\mathcal{N}(A + B)\) must be in \(\mathcal{N}(A)\), and \(\mathcal{N}(B)\).
this implies that $\dim \mathcal{N}(A + B)$ cannot be greater than $\dim \mathcal{N}(A)$ or $\dim \mathcal{N}(B)$, hence a positive combination of positive semidefinite matrices can only gain rank.

It follows that the conic hull of the set of rank-$k$ outer products is the set of positive semidefinite matrices of rank greater than or equal to $k$, along with the zero matrix.