EE364a Review Session 2

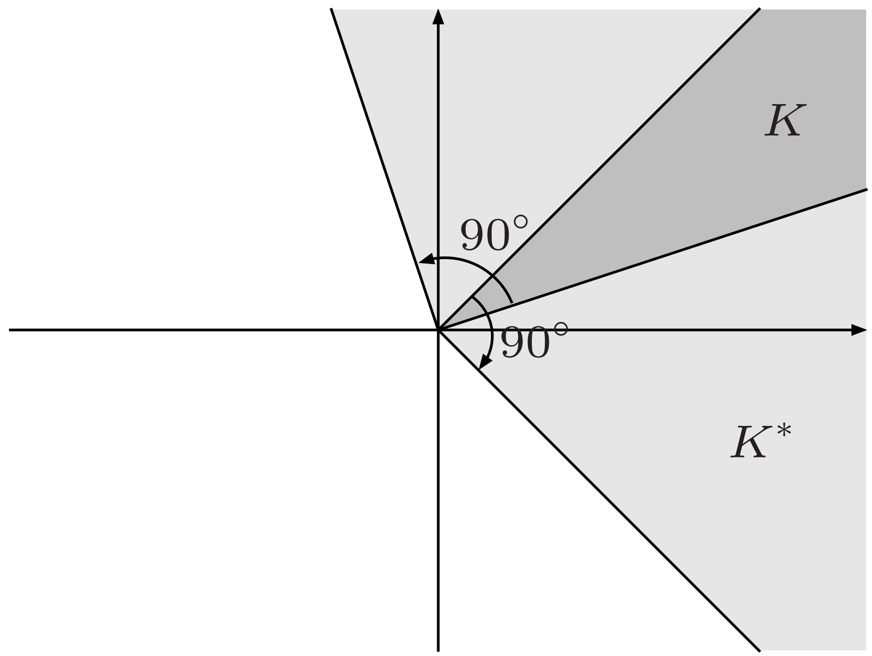
session outline:

- dual cones
- convex functions
- conjugate function

Dual cones

for a cone K, the dual cone is $K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$

$y \in K^*$ if and only if the halfspace $\{z \mid y^T z \geq 0\}$ contains K



ex. 2.32: Find the dual cone of $\{Ax \mid x \succeq 0\}$, where $A \in \mathbf{R}^{m \times n}$.

solution.

$$\begin{aligned} K^* &= \{y \mid y^T x \geq 0 \text{ for all } x \in K\} \\ &= \{y \mid (A^T y)^T x \geq 0 \text{ for all } x \succeq 0\} \end{aligned}$$

this is equivalent to

$$K^* = \{y \mid A^T y \succeq 0\}$$

- *sufficient*: $A^T y \succeq 0 \Rightarrow (A^T y)^T x \geq 0$ for all $x \succeq 0$

- *necessary*: assume that $(A^T y)_i < 0$ for some i.
  then $(A^T y)^T e_i < 0$, which is a contradiction.

Convex functions

- tools
  - definition of convexity
  - first-order condition
  - second-order condition
  - restriction to a line
  - simple examples (negative log, norms, quadratic-over-linear, log-sum-exp, . . .)
- convexity-preserving operations
  - nonnegative weighted sum
  - composition with an affine function
  - pointwise maximum and supremum
  - minimization (over convex sets)
  - composition
  - perspective

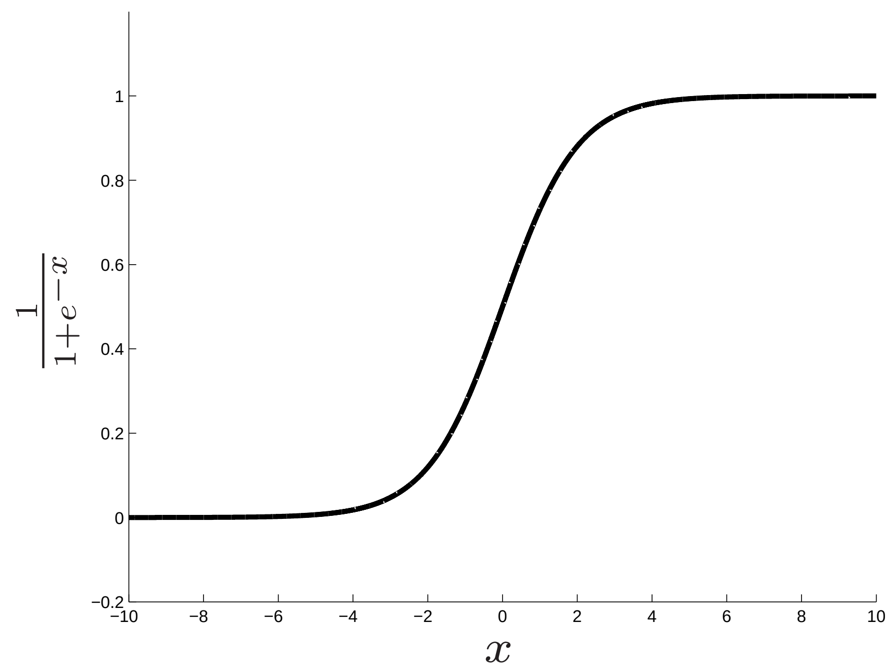
example: sigmoid / logistic function

$$f(x) = \frac{1}{1 + e^{-x}}$$

- is it convex? concave?
- is it quasiconvex? quasiconcave?
- is it log-convex? log-concave?

- is it convex? concave?

$$f(x) = \frac{1}{1+e^{-x}}$$



solution.

- by looking at the graph, it is neither convex nor concave.
- alternatively, $f''(x) = -\frac{e^{-x}(1-e^{-x})}{(1+e^{-x})^3} \begin{cases} > 0 & \text{if } x < 0 \\ \leq 0 & \text{if } x \geq 0 \end{cases}$

- is it quasiconvex? quasiconcave?

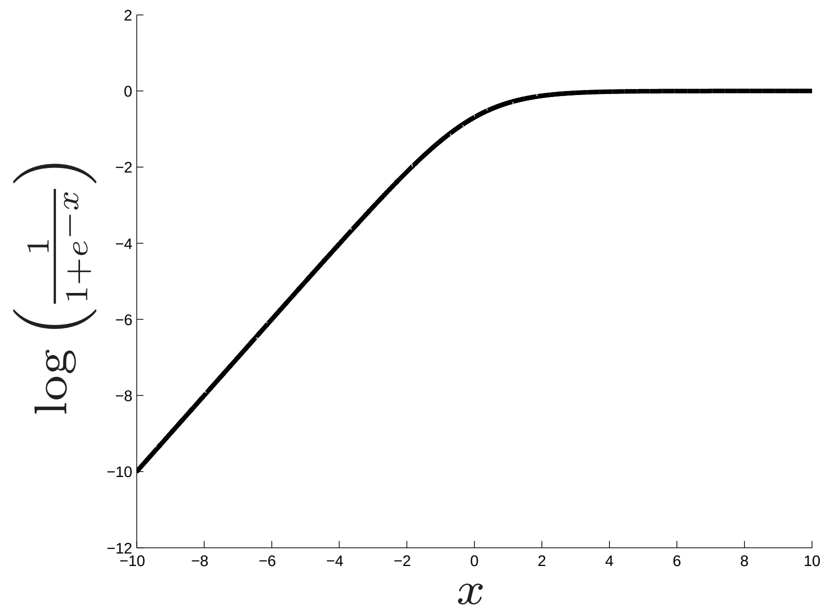
$$f(x) = \frac{1}{1 + e^{-x}}$$

solution.

- sublevel sets C_α are convex $\Rightarrow$ quasiconvex
  - for $\alpha \leq 0$, $C_\alpha = \emptyset$
  - for $\alpha \geq 1$, $C_\alpha = \mathbf{R}$
  - for $0 < \alpha < 1$, $C_\alpha = (-\infty, f^{-1}(\alpha)]$
- similarly, superlevel sets are convex $\Rightarrow$ quasiconcave
- for $x \in \mathbf{R}$, $f(x)$ monotonic $\Leftrightarrow$ quasiconvex and quasiconcave

- is it log-convex? log-concave?

$$f(x) = \frac{1}{1+e^{-x}}$$



solution.

– not log-convex
– is log-concave ($\log f(x)$ is negative of log-sum-exp, evaluated at $z_1 = 1$, $z_2 = -x$)

example: is the following a convex function (in $x, y, z \in \mathbf{R}$)?

$$f(x, y, z) = \frac{(x-z)^2}{y+1} + \max\left(1 + |x| - y, \frac{1}{\sqrt{z}}, 0\right)$$

(with domain $y + 1 > 0$, $z > 0$)

solution. The following steps show that the function is convex:

- $|x|$ is convex in x, and $1 - y$ is affine, so $1 + |x| - y$ is convex
- $\frac{1}{\sqrt{z}}$ is a negative-power function, so convex in z
- $\max$ term is convex, since its arguments are
- $\frac{(x-z)^2}{y+1}$ is composition of quadratic-over-linear functions $\frac{s^2}{t}$ with affine function that maps (x, y, z) to $(x - z, y + 1)$, so is convex
- sum of left and right terms is convex

Composition rules

composition of $g : \mathbf{R}^n \to \mathbf{R}^k$ and $h : \mathbf{R}^k \to \mathbf{R}$:

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \dots, g_k(x))$$

e.g., f is convex if g_i concave, h convex, $\tilde{h}$ nonincreasing in each argument

proof: (for $n = 1$, differentiable g, h)

$$f''(x) = g'(x)^T \underbrace{\nabla^2 h(g(x))}_{\succeq 0} g'(x) + \underbrace{\nabla h(g(x))}_{\preceq 0}^T \underbrace{g''(x)}_{\preceq 0}$$

ex. 3.22(b): Show that the following function is convex:

$$f(x,u,v) = -\sqrt{uv - x^T x}$$

on $\mathbf{dom}\, f = \{(x,u,v) \mid uv > x^T x,\ u,\ v > 0\}$. Use the fact that $x^T x/u$ is convex in (x,u) for $u > 0$, and that $-\sqrt{x_1 x_2}$ is convex on $\mathbf{R}^2_{++}$.

solution.

- take $f(x,u,v) = -\sqrt{u(v - x^T x/u)}$
- $g_1(u,v,x) = u$ and $g_2(u,v,x) = v - x^T x/u$ are concave
- the function

  $$h(z_1,z_2) = \begin{cases} -\sqrt{z_1 z_2} & \text{if } z \succeq 0 \\ 0 & \text{otherwise} \end{cases}$$

  is convex and decreasing in each argument
- $f(u,v,x) = h(g(u,v,x))$ is convex

Conjugate function

the **conjugate** of a function f is

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} (y^T x - f(x))$$

ex. 3.36(a): Derive the conjugate of the *max function*

$$f(x) = \max_{i=1,\ldots,n} x_i \text{ on } \mathbf{R}^n$$

solution (partial). we see what happens for $n = 2$

- first, want to determine the domain for y of the conjugate function $f^*(y)$ ($i.e.$, where $y^T x - f(x)$ is bounded above)
- try y with some $y_k < 0$:
  - e.g., choose $y = (-1, 0)$
  - then if $x = -te_1$, we have $y^T x - \max x_i = t - 0 \to \infty$ as $t \to \infty$
  - so $y \succeq 0$
- (continued on next slide. . .)

- now look at $y \succeq 0$:
  - try $y = (0.7, 0.7)$
  - then if $x = t\mathbf{1}$, we have $y^T x - \max x_i = t(\mathbf{1}^T y) - t = 1.4t - t \to \infty$ as $t \to \infty$
  - $y = (0.7, 0.7) \notin \mathbf{dom}\, f^*$
  - for $x = t\mathbf{1}$, if $y \succeq 0$, we need $\mathbf{1}^T y = 1$ for $y^T x - \max x_i$ to be bounded above

- for $y \in \{y \succeq 0 \mid \mathbf{1}^T y = 1\}$, what is

$$\sup_{x \in \mathbf{dom}\, f} (y^T x - \max_{i=1,\ldots,n} x_i)?$$

  - can show that $y^T x \leq \max x_i$ (why?), and equality holds when $x = 0$
  - so for $y \succeq 0$ and $\mathbf{1}^T y = 1$, the $\sup$ is always bounded above
  - thus,

$$f^*(y) = \begin{cases} 0 & \text{if } y \succeq 0 \text{ and } \mathbf{1}^T y = 1 \\ \infty & \text{otherwise} \end{cases}$$