Convex optimization examples

• force/moment generation with thrusters
• minimum-time optimal control
• optimal transmitter power allocation
• phased-array antenna beamforming
• optimal receiver location
• power allocation in FDM system
• optimizing structural dynamics
Force/moment generation with thrusters

- rigid body with center of mass at origin $p = 0 \in \mathbb{R}^2$

- $n$ forces with magnitude $u_i$, acting at $p_i = (p_{ix}, p_{iy})$, in direction $\theta_i$
• resulting horizontal force: \( F_x = \sum_{i=1}^{n} u_i \cos \theta_i \)

• resulting vertical force: \( F_y = \sum_{i=1}^{n} u_i \sin \theta_i \)

• resulting torque: \( T = \sum_{i=1}^{n} (p_{iy} u_i \cos \theta_i - p_{ix} u_i \sin \theta_i) \)

• force limits: \( 0 \leq u_i \leq 1 \) (thrusters)

• fuel usage: \( u_1 + \cdots + u_n \)

**Problem:** find thruster forces \( u_i \) that yield given desired forces and torques \( F_x^{\text{des}}, F_y^{\text{des}}, T^{\text{des}} \), and minimize fuel usage (if feasible)
can be expressed as LP:

\[
\begin{align*}
\text{minimize} & \quad 1^T u \\
\text{subject to} & \quad F u = f^{\text{des}} \\
& \quad 0 \leq u_i \leq 1, \quad i = 1, \ldots, n
\end{align*}
\]

where

\[
F = \begin{bmatrix}
\cos \theta_1 & \cdots & \cos \theta_n \\
\sin \theta_1 & \cdots & \sin \theta_n \\
p_{1y} \cos \theta_1 - p_{1x} \sin \theta_1 & \cdots & p_{ny} \cos \theta_n - p_{nx} \sin \theta_n
\end{bmatrix},
\]

\[
f^{\text{des}} = (F_x^{\text{des}}, F_y^{\text{des}}, T^{\text{des}}), \quad 1 = (1, 1, \cdots, 1)
\]
Extensions of thruster problem

• opposing thruster pairs:

\[
\begin{align*}
\text{minimize} & \quad \|u\|_1 = \sum_{i=1}^{n} |u_i| \\
\text{subject to} & \quad F u = f^{\text{des}} \\
& \quad |u_i| \leq 1, \quad i = 1, \ldots, n
\end{align*}
\]

can express as LP

• more accurate fuel use model:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{n} \phi_i(u_i) \\
\text{subject to} & \quad F u = f^{\text{des}} \\
& \quad 0 \leq u_i \leq 1, \quad i = 1, \ldots, n
\end{align*}
\]

\(\phi_i\) are piecewise linear increasing convex functions
can express as LP
• minimize maximum force/moment error:

\[
\begin{align*}
\text{minimize} & \quad \|F u - f^{\text{des}}\|_\infty \\
\text{subject to} & \quad 0 \leq u_i \leq 1, \quad i = 1, \ldots, n
\end{align*}
\]

can express as LP

• minimize number of thrusters used:

\[
\begin{align*}
\text{minimize} & \quad \# \text{ thrusters on} \\
\text{subject to} & \quad F u = f^{\text{des}} \\
& \quad 0 \leq u_i \leq 1, \quad i = 1, \ldots, n
\end{align*}
\]

can’t express as LP

(but we could check feasibility of each of the \(2^n\) subsets of thrusters)
Minimum-time optimal control

- linear dynamical system:

\[ x(t + 1) = Ax(t) + Bu(t), \quad t = 0, 1, \ldots, K, \quad x(0) = x_0 \]

- inputs limited to range \([-1, 1]\):

\[ \|u(t)\|_\infty \leq 1, \quad t = 0, 1, \ldots, K \]

- settling time:

\[ f(u(0), \ldots, u(K)) = \min \{ T \mid x(t) = 0 \text{ for } T \leq t \leq K + 1 \} \]
settling time $f$ is quasiconvex function of $(u(0), \ldots, u(K))$:

$$f(u(0), u(1), \ldots, u(K)) \leq T$$

if and only if for all $t = T, \ldots, K + 1$

$$x(t) = A^t x_0 + A^{t-1} Bu(0) + \cdots + Bu(t-1) = 0$$

i.e., sublevel sets are affine

**minimum-time optimal control problem:**

minimize \[ f(u(0), u(1), \ldots, u(K)) \]

subject to \[ \|u(t)\|_\infty \leq 1, \quad t = 0, \ldots, K \]

with variables $u(0), \ldots, u(K)$

a quasiconvex problem; can be solved via bisection with LPs
Minimum-time control example

three unit masses, connected by two unit springs with equilibrium length one

\[ u(t) \in \mathbb{R}^2 \text{ is force on left & right masses over time interval } (0.15t, 0.15(t + 1)] \]

\[ u_1(t) \quad u_2(t) \]

problem: pick \( u(0), \ldots, u(K) \) to bring masses to positions \((-1, 0, 1)\) (at rest), as quickly as possible, from initial condition \((-3, 0, 2)\) (at rest)
optimal solution:
Optimal transmitter power allocation

- $m$ transmitters, $mn$ receivers all at same frequency

- Transmitter $i$ wants to transmit to $n$ receivers labeled $(i, j)$, $j = 1, \ldots, n$

- $A_{ijk}$ is path gain from transmitter $k$ to receiver $(i, j)$

- $N_{ij}$ is (self) noise power of receiver $(i, j)$

- Variables: transmitter powers $p_k$, $k = 1, \ldots, m$
at receiver \((i,j)\):

- signal power:
  \[ S_{ij} = A_{iji}p_i \]

- noise plus interference power:
  \[ I_{ij} = \sum_{k \neq i} A_{ijk}p_k + N_{ij} \]

- signal to interference/noise ratio (SINR): \( S_{ij}/I_{ij} \)

**problem:** choose \(p_i\) to maximize smallest SINR:

\[
\begin{align*}
\text{maximize} & \quad \min_{i,j} \frac{A_{iji}p_i}{\sum_{k \neq i} A_{ijk}p_k + N_{ij}} \\
\text{subject to} & \quad 0 \leq p_i \leq p_{\text{max}}
\end{align*}
\]

... a (generalized) linear fractional program
Phased-array antenna beamforming

- omnidirectional antenna elements at positions \((x_1, y_1), \ldots, (x_n, y_n)\)

- unit plane wave incident from angle \(\theta\) induces in \(i\)th element a signal

\[
e^{j(x_i \cos \theta + y_i \sin \theta - \omega t)}
\]

\((j = \sqrt{-1}, \text{ frequency } \omega, \text{ wavelength } 2\pi)\)
• demodulate to get output $e^{j(x_i \cos \theta + y_i \sin \theta)} \in \mathbb{C}$

• linearly combine with complex weights $w_i$:

$$y(\theta) = \sum_{i=1}^{n} w_i e^{j(x_i \cos \theta + y_i \sin \theta)}$$

• $y(\theta)$ is (complex) \textit{antenna array gain pattern}

• $|y(\theta)|$ gives sensitivity of array as function of incident angle $\theta$

• depends on design variables $\text{Re } w, \text{Im } w$
  (called \textit{antenna array weights} or \textit{shading coefficients})

\textbf{design problem}: choose $w$ to achieve desired gain pattern
Sidelobe level minimization

make $|y(\theta)|$ small for $|\theta - \theta_{\text{tar}}| > \alpha$

($\theta_{\text{tar}}$: target direction; $2\alpha$: beamwidth)

via least-squares (discretize angles)

minimize $\sum_i |y(\theta_i)|^2$
subject to $y(\theta_{\text{tar}}) = 1$

(sum is over angles outside beam)

least-squares problem with two (real) linear equality constraints
\[ \theta_{\text{tar}} = 30^\circ \]

| \[ y(\theta) \] |

sidelobe level

50°

10°
**minimize sidelobe level** (discretize angles)

\[
\text{minimize } \max_i |y(\theta_i)| \\
\text{subject to } y(\theta_{\text{tar}}) = 1
\]

(max over angles outside beam)

can be cast as SOCP

\[
\text{minimize } t \\
\text{subject to } |y(\theta_i)| \leq t \\
y(\theta_{\text{tar}}) = 1
\]
$\theta_{\text{tar}} = 30^\circ$

$|y(\theta)|$

sidelobe level

$50^\circ$

$10^\circ$
Extensions

convex (& quasiconvex) extensions:

- \( y(\theta_0) = 0 \) (null in direction \( \theta_0 \))

- \( w \) is real (amplitude only shading)

- \( |w_i| \leq 1 \) (attenuation only shading)

- minimize \( \sigma^2 \sum_{i=1}^{n} |w_i|^2 \) (thermal noise power in \( y \))

- minimize beamwidth given a maximum sidelobe level

nonconvex extension:

- maximize number of zero weights
Optimal receiver location

- $N$ transmitter frequencies $1, \ldots, N$
- transmitters at locations $a_i, b_i \in \mathbb{R}^2$ use frequency $i$
- transmitters at $a_1, a_2, \ldots, a_N$ are the wanted ones
- transmitters at $b_1, b_2, \ldots, b_N$ are interfering
- receiver at position $x \in \mathbb{R}^2$
• (signal) receiver power from $a_i$: $\|x - a_i\|^{-\alpha}$ ($\alpha \approx 2.1$)

• (interfering) receiver power from $b_i$: $\|x - b_i\|^{-\alpha}$ ($\alpha \approx 2.1$)

• worst signal to interference ratio, over all frequencies, is

$$S/I = \min_i \frac{\|x - a_i\|^{-\alpha}}{\|x - b_i\|^{-\alpha}}$$

• what receiver location $x$ maximizes $S/I$?
S/I is quasiconcave on \( \{x \mid S/I \geq 1\} \), i.e., on

\[
\{x \mid \|x - a_i\| \leq \|x - b_i\|, \quad i = 1, \ldots, N\}
\]

can use bisection; every iteration is a convex quadratic feasibility problem.
Power allocation in FDM system

frequency division multiplex (FDM) system

- signal $u_i$ modulates carrier frequency $f_i$ with power $p_i$
- channel is slightly nonlinear
- powers affect signal power, interference power at each $y_i$
- problem: choose powers to maximize minimum SINR (signal to noise & interference ratio)
• demodulated signal power in $y_i$ proportional to $p_i$

• noise power in $y_i$ is $\sigma_i^2$

• interference power in $y_i$ is sum of crosstalk & intermodulation products from nonlinearity

• crosstalk power $c_i$ is linear in powers:

$$c = C'p, \quad C_{ij} \geq 0$$

$C'$ is often tridiagonal, i.e., have crosstalk from adjacent channels only
- intermodulation power: \( k \)th order IM products have frequencies

\[
\pm f_{i_1} \pm f_{i_2} \pm \cdots \pm f_{i_k}
\]

with power proportional to \( p_{i_1} p_{i_2} \cdots p_{i_k} \)
e.g., for frequencies 1, 2, 3:

<table>
<thead>
<tr>
<th>frequency</th>
<th>IM product</th>
<th>pwr. prop. to</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 + 1</td>
<td>( p_1^2 )</td>
</tr>
<tr>
<td>3</td>
<td>1 + 2</td>
<td>( p_1 p_2 )</td>
</tr>
<tr>
<td>1</td>
<td>2 − 1</td>
<td>( p_2 p_1 )</td>
</tr>
<tr>
<td>1</td>
<td>3 − 2</td>
<td>( p_3 p_2 )</td>
</tr>
<tr>
<td>2</td>
<td>3 − 1</td>
<td>( p_3 p_1 )</td>
</tr>
<tr>
<td>3</td>
<td>1 + 1 + 1</td>
<td>( p_1^3 )</td>
</tr>
<tr>
<td>1</td>
<td>1 + 1 − 1</td>
<td>( p_1^3 )</td>
</tr>
<tr>
<td>2</td>
<td>2 + 1 − 1</td>
<td>( p_2 p_1^2 )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

- total IM power at \( f_i \) is (complicated) polynomial of \( p_1, \ldots, p_n \), with nonnegative coefficients

Convex optimization examples
inverse SINR at frequency $i$

\[
\frac{\text{noise + crosstalk + IM power}}{\text{signal power}}
\]

is posynomial function of $p_1, \ldots, p_n$

hence, problem such as

\[
\begin{align*}
\text{maximize} & \quad \min_i \text{SINR}_i \\
\text{subject to} & \quad 0 < p_i \leq P_{\text{max}}
\end{align*}
\]

is geometric program
Optimizing structural dynamics

linear elastic structure

\[ M \ddot{d} + K d = 0 \]

- \( d(t) \in \mathbb{R}^k \): vector of displacements
- \( M = M^T \succ 0 \) is mass matrix; \( K = K^T \succ 0 \) is stiffness matrix
Fundamental frequency

- solutions have form

\[ d_i(t) = \sum_{j=1}^{k} \alpha_{ij} \cos(\omega_j t - \phi_j) \]

where \( 0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_k \) are the modal frequencies, i.e., positive solutions of \( \det(\omega^2 M - K) = 0 \)

- fundamental frequency:

\[ \omega_1 = \lambda^{1/2}_{\min}(K, M) = \lambda^{1/2}_{\min}(M^{-1/2}KM^{-1/2}) \]

- structure behaves like mass at frequencies below \( \omega_1 \)
- gives stiffness measure (the larger \( \omega_1 \), the stiffer the structure)

- \( \omega_1 \geq \Omega \iff \Omega^2 M - K \preceq 0 \) so \( \omega_1 \) is quasiconcave function of \( M, K \)
• design variables: \( x_i \), cross-sectional area of structural member \( i \) (geometry of structure fixed)

\[ M(x) = M_0 + \sum_i x_i M_i, \quad K(x) = K_0 + \sum_i x_i K_i \]

• structure weight \( w = w_0 + \sum_i x_i w_i \)

• problem: minimize weight s.t. \( \omega_1 \geq \Omega \), limits on cross-sectional areas as SDP:

\[
\begin{align*}
\text{minimize} \quad & w_0 + \sum_i x_i w_i \\
\text{subject to} \quad & \Omega^2 M(x) - K(x) \preceq 0 \\
& l_i \leq x_i \leq u_i
\end{align*}
\]