ℓ_1-norm Methods for Convex-Cardinality Problems

- problems involving cardinality
- the ℓ_1-norm heuristic
- convex relaxation and convex envelope interpretations
- examples
- recent results

ℓ_1-norm heuristics for cardinality problems

- cardinality problems arise often, but are hard to solve exactly
- a simple heuristic, that relies on ℓ_1-norm, seems to work well
- used for many years, in many fields
  - sparse design
  - LASSO, robust estimation in statistics
  - support vector machine (SVM) in machine learning
  - total variation reconstruction in signal processing, geophysics
  - compressed sensing
- new theoretical results guarantee the method works, at least for a few problems

Cardinality

- the **cardinality** of $x \in \mathbf{R}^n$, denoted $\mathbf{card}(x)$, is the number of nonzero components of x
- $\mathbf{card}$ is separable; for scalar x, $\mathbf{card}(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$
- $\mathbf{card}$ is quasiconcave on $\mathbf{R}^n_+$ (but not $\mathbf{R}^n$) since

$$\mathbf{card}(x+y) \geq \min\{\mathbf{card}(x), \mathbf{card}(y)\}$$

  holds for $x, y \succeq 0$

- but otherwise has no convexity properties
- arises in many problems

General convex-cardinality problems

a **convex-cardinality problem** is one that would be convex, except for appearance of $\mathbf{card}$ in objective or constraints

examples (with $\mathcal{C}$, f convex):

- convex minimum cardinality problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

- convex problem with cardinality constraint:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \mathbf{card}(x) \leq k \end{array}$$

Solving convex-cardinality problems

convex-cardinality problem with $x \in \mathbf{R}^n$

- if we fix the sparsity pattern of x ($i.e.$, which entries are zero/nonzero) we get a convex problem

- by solving 2^n convex problems associated with all possible sparsity patterns, we can solve convex-cardinality problem
(possibly practical for $n \leq 10$; not practical for $n > 15$ or so . . .)

- general convex-cardinality problem is (NP-) hard

- can solve globally by branch-and-bound
  - can work for particular problem instances (with some luck)
  - in worst case reduces to checking all (or many of) 2^n sparsity patterns

Boolean LP as convex-cardinality problem

- Boolean LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \quad x_i \in \{0,1\} \end{array}$$

  includes many famous (hard) problems, *e.g.*, 3-SAT, traveling salesman

- can be expressed as

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \quad \mathbf{card}(x) + \mathbf{card}(1-x) \leq n \end{array}$$

  since $\mathbf{card}(x) + \mathbf{card}(1-x) \leq n \iff x_i \in \{0,1\}$

- conclusion: general convex-cardinality problem is hard

Sparse design

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

- find sparsest design vector x that satisfies a set of specifications
- zero values of x simplify design, or correspond to components that aren't even needed
- examples:
  - FIR filter design (zero coefficients reduce required hardware)
  - antenna array beamforming (zero coefficients correspond to unneeded antenna elements)
  - truss design (zero coefficients correspond to bars that are not needed)
  - wire sizing (zero coefficients correspond to wires that are not needed)

Sparse modeling / regressor selection

fit vector $b \in \mathbf{R}^m$ as a linear combination of k regressors (chosen from n possible regressors)

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

- gives k-term model
- chooses subset of k regressors that (together) best fit or explain b
- can solve (in principle) by trying all $\binom{n}{k}$ choices
- variations:
  - minimize $\mathbf{card}(x)$ subject to $\|Ax - b\|_2 \leq \epsilon$
  - minimize $\|Ax - b\|_2 + \lambda\,\mathbf{card}(x)$

Sparse signal reconstruction

- estimate signal x, given
  - noisy measurement $y = Ax + v$, $v \sim \mathcal{N}(0, \sigma^2 I)$ (A is known; v is not)
  - prior information $\mathbf{card}(x) \leq k$

- maximum likelihood estimate $\hat{x}_{\mathrm{ml}}$ is solution of

$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

Estimation with outliers

- we have measurements $y_i = a_i^T x + v_i + w_i$, $i = 1, \ldots, m$
- noises $v_i \sim \mathcal{N}(0, \sigma^2)$ are independent
- only assumption on w is sparsity: $\mathbf{card}(w) \leq k$
- $\mathcal{B} = \{i \mid w_i \neq 0\}$ is set of bad measurements or *outliers*
- maximum likelihood estimate of x found by solving

$$\begin{array}{ll} \text{minimize} & \sum_{i \notin \mathcal{B}} (y_i - a_i^T x)^2 \\ \text{subject to} & |\mathcal{B}| \leq k \end{array}$$

  with variables x and $\mathcal{B} \subseteq \{1, \ldots, m\}$
- equivalent to

$$\begin{array}{ll} \text{minimize} & \|y - Ax - w\|_2^2 \\ \text{subject to} & \mathbf{card}(w) \leq k \end{array}$$

Minimum number of violations

- set of convex inequalities

$$f_1(x) \le 0, \; \ldots, \; f_m(x) \le 0, \qquad x \in \mathcal{C}$$

- choose x to minimize the number of violated inequalities:

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(t) \\ \text{subject to} & f_i(x) \le t_i, \quad i = 1, \ldots, m \\ & x \in \mathcal{C}, \quad t \ge 0 \end{array}$$

- determining whether zero inequalities can be violated is (easy) convex feasibility problem

Linear classifier with fewest errors

- given data $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbf{R}^n \times \{-1, 1\}$
- we seek linear (affine) classifier $y \approx \mathbf{sign}(w^T x + v)$
- classification error corresponds to $y_i(w^T x + v) \leq 0$
- to find w, v that give fewest classification errors:

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(t) \\ \text{subject to} & y_i(w^T x_i + v) + t_i \geq 1, \quad i = 1, \ldots, m \end{array}$$

with variables w, v, t (we use homogeneity in w, v here)

Smallest set of mutually infeasible inequalities

- given a set of mutually infeasible convex inequalities $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

- find smallest (cardinality) subset of these that is infeasible

- certificate of infeasibility is $g(\lambda) = \inf_x(\sum_{i=1}^m \lambda_i f_i(x)) \geq 1$, $\lambda \succeq 0$

- to find smallest cardinality infeasible subset, we solve

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(\lambda) \\ \text{subject to} & g(\lambda) \geq 1, \quad \lambda \succeq 0 \end{array}$$

(assuming some constraint qualifications)

Portfolio investment with linear and fixed costs

- we use budget B to purchase (dollar) amount $x_i \geq 0$ of stock i
- trading fee is fixed cost plus linear cost: $\beta\,\mathbf{card}(x) + \alpha^T x$
- budget constraint is $\mathbf{1}^T x + \beta\,\mathbf{card}(x) + \alpha^T x \leq B$
- mean return on investment is $\mu^T x$; variance is $x^T \Sigma x$
- minimize investment variance (risk) with mean return $\geq R_{\min}$:

$$
\begin{array}{ll}
\text{minimize} & x^T \Sigma x \\
\text{subject to} & \mu^T x \geq R_{\min}, \quad x \succeq 0 \\
& \mathbf{1}^T x + \beta\,\mathbf{card}(x) + \alpha^T x \leq B
\end{array}
$$

Piecewise constant fitting

- fit corrupted $x_{\rm cor}$ by a piecewise constant signal $\hat{x}$ with k or fewer jumps
- problem is convex once location (indices) of jumps are fixed
- $\hat{x}$ is piecewise constant with $\leq k$ jumps $\Longleftrightarrow \mathbf{card}(D\hat{x}) \leq k$, where

$$D = \left[\begin{array}{cccccc} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 \end{array}\right] \in \mathbf{R}^{(n-1)\times n}$$

- as convex-cardinality problem:

$$\begin{array}{ll} \mbox{minimize} & \|\hat{x} - x_{\rm cor}\|_2 \\ \mbox{subject to} & \mathbf{card}(D\hat{x}) \leq k \end{array}$$

Piecewise linear fitting

- fit $x_{\rm cor}$ by a piecewise linear signal $\hat{x}$ with k or fewer kinks

- as convex-cardinality problem:

$$\begin{array}{ll} \mbox{minimize} & \|\hat{x} - x_{\rm cor}\|_2 \\ \mbox{subject to} & \mathbf{card}(\nabla \hat{x}) \leq k \end{array}$$

where

$$\nabla = \left[\begin{array}{cccccc} -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \end{array} \right]$$

ℓ_1-norm heuristic

- replace $\mathbf{card}(z)$ with $\gamma\|z\|_1$, or add regularization term $\gamma\|z\|_1$ to objective

- $\gamma > 0$ is parameter used to achieve desired sparsity (when $\mathbf{card}$ appears in constraint, or as term in objective)

- more sophisticated versions use $\sum_i w_i|z_i|$ or $\sum_i w_i(z_i)_+ + \sum_i v_i(z_i)_-$, where w, v are positive weights

Example: Minimum cardinality problem

- start with (hard) minimum cardinality problem

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

  ($\mathcal{C}$ convex)

- apply heuristic to get (easy) ℓ_1-norm minimization problem

$$\begin{array}{ll} \text{minimize} & \|x\|_1 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Example: Cardinality constrained problem

- start with (hard) cardinality constrained problem (f, $\mathcal{C}$ convex)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \mathbf{card}(x) \leq k \end{array}$$

- apply heuristic to get (easy) ℓ_1-constrained problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \|x\|_1 \leq \beta \end{array}$$

or ℓ_1-regularized problem

$$\begin{array}{ll} \text{minimize} & f(x) + \gamma \|x\|_1 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

β, γ adjusted so that $\mathbf{card}(x) \leq k$

Polishing

- use ℓ_1 heuristic to find $\hat{x}$ with required sparsity
- fix the sparsity pattern of $\hat{x}$
- re-solve the (convex) optimization problem with this sparsity pattern to obtain final (heuristic) solution

Interpretation as convex relaxation

- start with

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \|x\|_\infty \leq R \end{array}$$

- equivalent to mixed Boolean convex problem

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T z \\ \text{subject to} & |x_i| \leq R z_i, \quad i = 1, \ldots, n \\ & x \in \mathcal{C}, \quad z_i \in \{0, 1\}, \quad i = 1, \ldots, n \end{array}$$

with variables x, z

- now relax $z_i \in \{0,1\}$ to $z_i \in [0,1]$ to obtain

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T z \\ \text{subject to} & |x_i| \leq R z_i, \quad i=1,\ldots,n \\ & x \in \mathcal{C} \\ & 0 \leq z_i \leq 1, \quad i=1,\ldots,n \end{array}$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & (1/R)\|x\|_1 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

the ℓ_1 heuristic

- optimal value of this problem is lower bound on original problem

Interpretation via convex envelope

- convex envelope f^{env} of a function f on set $\mathcal{C}$ is the largest convex function that is an underestimator of f on $\mathcal{C}$
- $\mathbf{epi}(f^{\mathrm{env}}) = \mathbf{Co}(\mathbf{epi}(f))$
- $f^{\mathrm{env}} = (f^*)^*$ (with some technical conditions)
- for x scalar, $|x|$ is the convex envelope of $\mathbf{card}(x)$ on $[-1, 1]$
- for $x \in \mathbf{R}^n$ scalar, $(1/R)\|x\|_1$ is convex envelope of $\mathbf{card}(x)$ on $\{z \mid \|z\|_\infty \leq R\}$

Weighted and asymmetric ℓ_1 heuristics

- minimize $\mathbf{card}(x)$ over convex set $\mathcal{C}$
- suppose we know lower and upper bounds on x_i over $\mathcal{C}$

$$x \in \mathcal{C} \implies l_i \leq x_i \leq u_i$$

  (best values for these can be found by solving $2n$ convex problems)
- if $u_i < 0$ or $l_i > 0$, then $\mathbf{card}(x_i) = 1$ (*i.e.*, $x_i \neq 0$) for all $x \in \mathcal{C}$
- assuming $l_i < 0$, $u_i > 0$, convex relaxation and convex envelope interpretations suggest using

$$\sum_{i=1}^{n} \left(\frac{(x_i)_+}{u_i} + \frac{(x_i)_-}{-l_i} \right)$$

  as surrogate (and also lower bound) for $\mathbf{card}(x)$

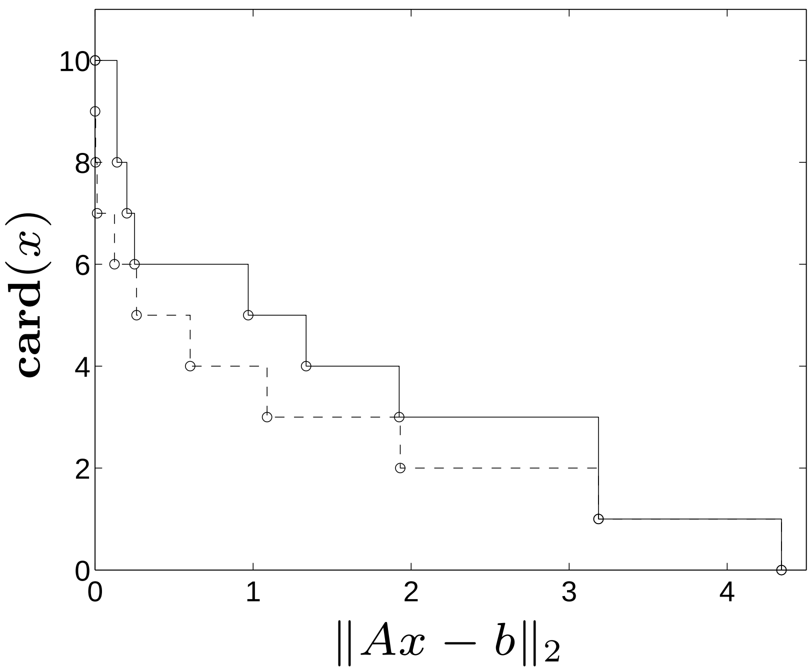
Regressor selection

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

- heuristic:
  - minimize $\|Ax - b\|_2 + \gamma \|x\|_1$
  - find smallest value of γ that gives $\mathbf{card}(x) \leq k$
  - fix associated sparsity pattern (*i.e.*, subset of selected regressors) and find x that minimizes $\|Ax - b\|_2$

Example (6.4 in BV book)

- $A \in \mathbf{R}^{10 \times 20}$, $x \in \mathbf{R}^{20}$, $b \in \mathbf{R}^{10}$
- dashed curve: exact optimal (via enumeration)
- solid curve: ℓ_1 heuristic with polishing



Sparse signal reconstruction

- convex-cardinality problem:

$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

- ℓ_1 heuristic:

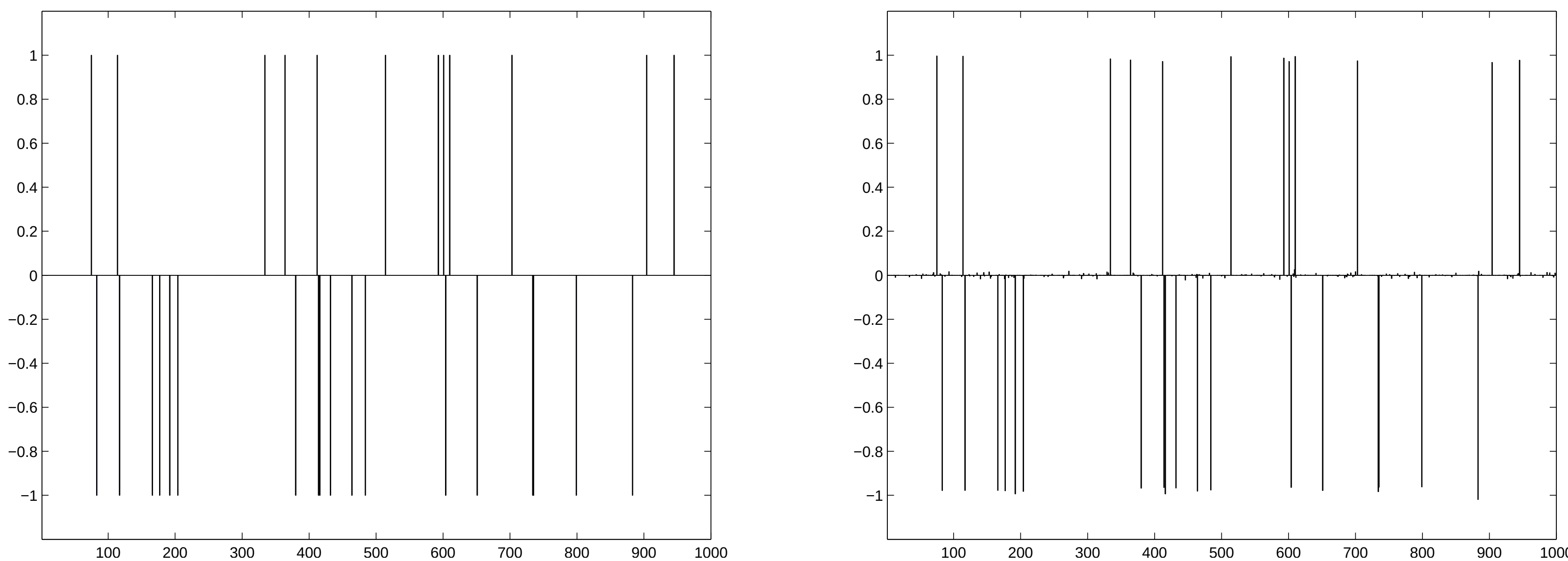
$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2 \\ \text{subject to} & \|x\|_1 \leq \beta \end{array}$$

  (called LASSO)

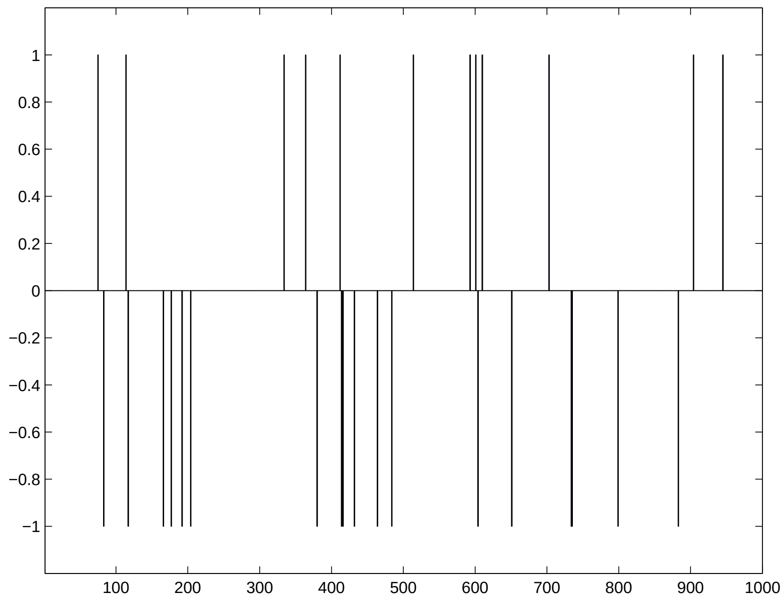
- another form: minimize $\|Ax - y\|_2 + \gamma\|x\|_1$
  (called basis pursuit denoising)

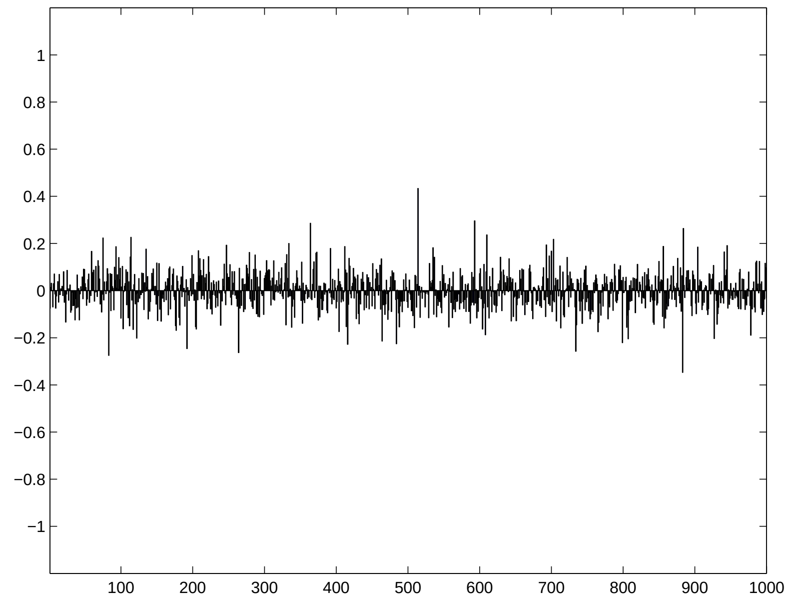
Example

- signal $x \in \mathbf{R}^n$ with $n = 1000$, $\mathbf{card}(x) = 30$
- $m = 200$ (random) noisy measurements: $y = Ax + v$, $v \sim \mathcal{N}(0, \sigma^2 \mathbf{1})$, $A_{ij} \sim \mathcal{N}(0, 1)$
- *left*: original; *right*: ℓ_1 reconstruction with $\gamma = 10^{-3}$



- ℓ_2 reconstruction; minimizes $\|Ax - y\|_2 + \gamma\|x\|_2$, where $\gamma = 10^{-3}$
- *left*: original; *right*: ℓ_2 reconstruction





Some recent theoretical results

- suppose $y = Ax$, $A \in \mathbf{R}^{m \times n}$, $\mathbf{card}(x) \leq k$
- to reconstruct x, clearly need $m \geq k$
- if $m \geq n$ and A is full rank, we can reconstruct x without cardinality assumption
- when does the ℓ_1 heuristic (minimizing $\|x\|_1$ subject to $Ax = y$) reconstruct x (exactly)?

recent results by Candès, Donoho, Romberg, Tao, . . .

- (for some choices of A) if $m \geq (C \log n)k$, ℓ_1 heuristic reconstructs x exactly, with overwhelming probability

- C is absolute constant; valid A's include
  - $A_{ij} \sim \mathcal{N}(0, \sigma^2)$
  - Ax gives Fourier transform of x at m frequencies, chosen from uniform distribution