EE364b Homework 1

1. For each of the following convex functions, explain how to calculate a subgradient at a given x.

   (a) $f(x) = \max_{i=1,\dots,m}(a_i^T x + b_i)$.

   (b) $f(x) = \max_{i=1,\dots,m} |a_i^T x + b_i|$.

   (c) $f(x) = \sup_{0 \le t \le 1} p(t)$, where $p(t) = x_1 + x_2 t + \cdots + x_n t^{n-1}$.

   (d) $f(x) = x_{[1]} + \cdots + x_{[k]}$, where $x_{[i]}$ denotes the ith largest element of the vector x.

   (e) $f(x) = \inf_{Ay \preceq b} \|x - y\|_2^2$, *i.e.*, the square of the Euclidean distance of x to the polyhedron defined by $Ay \preceq b$. You may assume that the inequalities $Ay \preceq b$ are strictly feasible.

   (f) $f(x) = \sup_{Ay \preceq b} y^T x$. (You can assume that the polyhedron defined by $Ay \preceq b$ is bounded.)

   Solution.

   (a) Find $k \in \{1, \dots, m\}$ for which $f(x) = a_k^T x + b_k$. Then $g = a_k$ is a subgradient at x.

   (b) Find $k \in \{1, \dots, m\}$ for which $f(x) = |a_k^T x + b_k|$. If $a_k^T x + b_k \ge 0$, then $g = a_k$ is a subgradient. If $a_k^T x + b_k \le 0$, then $g = -a_k$ is a subgradient.

   (c) Find $\hat{t} \in [0, 1]$ such that $f(x) = p(\hat{t})$ (for example, by plotting $p(t)$, or by computing all roots of the polynomial $p'(t)$, and selecting the real root at which $p(t)$ is maximum). Then $g = (1, \hat{t}, \dots, \hat{t}^{n-1})$ is a subgradient at x.

   (d) Sort the entries of x to find $x_{[1]}, \dots, x_{[n]}$. Let $\mathcal{I} = \{i_1, i_2, \dots, i_k\}$ be the set of indices associated with the k largest elements in x. Then take g to be a vector with $g_i = 1$ if $i \in \mathcal{I}$, and $g_i = 0$ otherwise.

   (e) We evaluate $f(x)$ by solving the optimization problem

   $$\begin{array}{ll} \text{minimize} & \|x - y\|_2^2 \\ \text{subject to} & Ay \preceq b \end{array}$$

   with variable y. The dual of this problem is

   $$\begin{array}{ll} \text{maximize} & -\frac{1}{4} z^T A A^T z + z^T A x - b^T z \\ \text{subject to} & z \succeq 0. \end{array}$$

   By Slater's condition, we have strong duality and the dual optimum is attained. (In fact we don't even need strict feasibility, because the constraints are linear.)

Let $z^\star$ be the optimal dual solution for the value of x at which we want a subgradient, *i.e.*, $z^\star \succeq 0$ and

$$f(x) = -\frac{1}{4} z^{\star T} A A^T z^\star + z^{\star T} A x - b^T z^\star.$$

By weak duality we have, for any $\hat{x}$,

$$\begin{aligned} f(\hat{x}) &\geq -\frac{1}{4} z^{\star T} A A^T z^\star + z^{\star T} A \hat{x} - b^T z^\star \\ &= f(x) + (A^T z^\star)^T (\hat{x} - x). \end{aligned}$$

By definition of the subgradient, this means that $A^T z^\star$ is a subgradient at x.

The KKT conditions for $y^\star$ to be an optimal point of the primal problem give:

$$A^T z^\star = 2(x - y^\star).$$

Therefore $(x - y^\star)$ is a subgradient at x.

(f) The set $\{y \mid Ay \preceq b\}$ is closed and bounded, hence compact. This means that the supremum in the definition of $f(x)$ is attained. Let $\hat{y} \in \{y \mid Ay \preceq b\}$ be the value of y for which $f(x) = \hat{y}^T x$. Then $\hat{y}$ is a subgradient of f at x.

2. *A convex function that is not subdifferentiable.* Verify that the following function $f : \mathbf{R} \to \mathbf{R}$ is convex, but not subdifferentiable at $x = 0$:

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0, \end{cases}$$

with $\mathbf{dom}\, f = \mathbf{R}_+$.

Solution. The function $f(x) = 0$ for $x > 0$, $f(x) = 1$ for $x = 0$, with domain $[0, \infty)$ is not subdifferentiable at $x = 0$.

We are going to prove this by contradiction. Suppose that f is subdifferentiable at $x = 0$ and let $g \neq 0$ be a subgradient of f at 0. The following must hold for all $x \geq 0$

$$f(x) \geq f(0) + gx.$$

This inequality holds for $x = 0$ (regardless of what g is). For $x > 0$, the inequality becomes

$$0 \geq 1 + gx.$$

In other words, we need to find a g such that $g \leq -1/x$ for all $x > 0$. This is obviously impossible—with arbitrarily small positive x, we have an arbitrarily large bound on the size of g.