Lecture 3
Linear algebra review

- vector space, subspaces
- independence, basis, dimension
- range, nullspace, rank
- change of coordinates
- norm, angle, inner product

Vector spaces

a *vector space* or *linear space* (over the reals) consists of

- a set $\mathcal{V}$
- a vector sum $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$
- a scalar multiplication : $\mathbf{R} \times \mathcal{V} \to \mathcal{V}$
- a distinguished element $0 \in \mathcal{V}$

which satisfy a list of properties

- $x + y = y + x, \quad \forall x, y \in \mathcal{V}$ (+ is commutative)

- $(x + y) + z = x + (y + z), \quad \forall x, y, z \in \mathcal{V}$ (+ is associative)

- $0 + x = x$, $\forall x \in \mathcal{V}$ (0 is additive identity)

- $\forall x \in \mathcal{V}\ \ \exists(-x) \in \mathcal{V}$ s.t. $x + (-x) = 0$ (existence of additive inverse)

- $(\alpha\beta)x = \alpha(\beta x), \quad \forall \alpha, \beta \in \mathbf{R} \quad \forall x \in \mathcal{V}$ (scalar mult. is associative)

- $\alpha(x + y) = \alpha x + \alpha y, \quad \forall \alpha \in \mathbf{R}\ \ \forall x, y \in \mathcal{V}$ (right distributive rule)

- $(\alpha + \beta)x = \alpha x + \beta x, \quad \forall \alpha, \beta \in \mathbf{R} \quad \forall x \in \mathcal{V}$ (left distributive rule)

- $1x = x, \quad \forall x \in \mathcal{V}$

Examples

- $\mathcal{V}_1 = \mathbf{R}^n$, with standard (componentwise) vector addition and scalar multiplication

- $\mathcal{V}_2 = \{0\}$ (where $0 \in \mathbf{R}^n$)

- $\mathcal{V}_3 = \mathrm{span}(v_1, v_2, \ldots, v_k)$ where

$$\mathrm{span}(v_1, v_2, \ldots, v_k) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_i \in \mathbf{R}\}$$

  and $v_1, \ldots, v_k \in \mathbf{R}^n$

Subspaces

- a *subspace* of a vector space is a *subset* of a vector space which is itself a vector space
- roughly speaking, a subspace is closed under vector addition and scalar multiplication
- examples $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ above are subspaces of $\mathbf{R}^n$

Vector spaces of functions

- $\mathcal{V}_4 = \{x : \mathbf{R}_+ \to \mathbf{R}^n \mid x \text{ is differentiable}\}$, where vector sum is sum of functions:
$$(x + z)(t) = x(t) + z(t)$$
and scalar multiplication is defined by
$$(\alpha x)(t) = \alpha x(t)$$
(a *point* in $\mathcal{V}_4$ is a *trajectory* in $\mathbf{R}^n$)

- $\mathcal{V}_5 = \{x \in \mathcal{V}_4 \mid \dot{x} = Ax\}$
(*points* in $\mathcal{V}_5$ are *trajectories* of the linear system $\dot{x} = Ax$)

- $\mathcal{V}_5$ is a subspace of $\mathcal{V}_4$

Independent set of vectors

a set of vectors $\{v_1, v_2, \ldots, v_k\}$ is *independent* if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0 \Longrightarrow \alpha_1 = \alpha_2 = \cdots = 0$$

some equivalent conditions:

- coefficients of $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$ are uniquely determined, $i.e.$,

  $$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$$

  implies $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k$

- no vector v_i can be expressed as a linear combination of the other vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$

Basis and dimension

set of vectors $\{v_1, v_2, \ldots, v_k\}$ is a *basis* for a vector space $\mathcal{V}$ if

- $v_1, v_2, \ldots, v_k$ span $\mathcal{V}$, *i.e.*, $\mathcal{V} = \operatorname{span}(v_1, v_2, \ldots, v_k)$
- $\{v_1, v_2, \ldots, v_k\}$ is independent

equivalent: every $v \in \mathcal{V}$ can be uniquely expressed as

$$v = \alpha_1 v_1 + \cdots + \alpha_k v_k$$

fact: for a given vector space $\mathcal{V}$, the number of vectors in any basis is the same

number of vectors in any basis is called the *dimension* of $\mathcal{V}$, denoted $\mathbf{dim}\mathcal{V}$

(we assign $\mathbf{dim}\{0\} = 0$, and $\mathbf{dim}\mathcal{V} = \infty$ if there is no basis)

Nullspace of a matrix

the *nullspace* of $A \in \mathbf{R}^{m\times n}$ is defined as

$$\mathcal{N}(A) = \{\ x \in \mathbf{R}^n \mid Ax = 0\ \}$$

- $\mathcal{N}(A)$ is set of vectors mapped to zero by $y = Ax$
- $\mathcal{N}(A)$ is set of vectors orthogonal to all rows of A

$\mathcal{N}(A)$ gives *ambiguity* in x given $y = Ax$:

- if $y = Ax$ and $z \in \mathcal{N}(A)$, then $y = A(x+z)$
- conversely, if $y = Ax$ and $y = A\tilde{x}$, then $\tilde{x} = x + z$ for some $z \in \mathcal{N}(A)$

Zero nullspace

A is called *one-to-one* if 0 is the only element of its nullspace:
$\mathcal{N}(A) = \{0\} \Longleftrightarrow$

- x can always be uniquely determined from $y = Ax$
  ($i.e.$, the linear transformation $y = Ax$ doesn't 'lose' information)
- mapping from x to Ax is one-to-one: different x's map to different y's
- columns of A are independent (hence, a basis for their span)
- A has a *left inverse*, $i.e.$, there is a matrix $B \in \mathbf{R}^{n \times m}$ s.t. $BA = I$
- $\det(A^T A) \neq 0$

(we'll establish these later)

Interpretations of nullspace

suppose $z \in \mathcal{N}(A)$

$y = Ax$ represents **measurement** of x

- z is undetectable from sensors — get zero sensor readings
- x and $x + z$ are indistinguishable from sensors: $Ax = A(x + z)$

$\mathcal{N}(A)$ characterizes *ambiguity* in x from measurement $y = Ax$

$y = Ax$ represents **output** resulting from input x

- z is an input with no result
- x and $x + z$ have same result

$\mathcal{N}(A)$ characterizes *freedom of input choice* for given result

Range of a matrix

the *range* of $A \in \mathbf{R}^{m\times n}$ is defined as

$$\mathcal{R}(A) = \{Ax \mid x \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$$

$\mathcal{R}(A)$ can be interpreted as

- the set of vectors that can be 'hit' by linear mapping $y = Ax$
- the span of columns of A
- the set of vectors y for which $Ax = y$ has a solution

Onto matrices

A is called *onto* if $\mathcal{R}(A) = \mathbf{R}^m \iff$

- $Ax = y$ can be solved in x for any y
- columns of A span $\mathbf{R}^m$
- A has a *right inverse*, $i.e.$, there is a matrix $B \in \mathbf{R}^{n \times m}$ s.t. $AB = I$
- rows of A are independent
- $\mathcal{N}(A^T) = \{0\}$
- $\det(AA^T) \neq 0$

(some of these are not obvious; we'll establish them later)

Interpretations of range

suppose $v \in \mathcal{R}(A)$, $w \notin \mathcal{R}(A)$

$y = Ax$ represents **measurement** of x

- $y = v$ is a *possible* or *consistent* sensor signal
- $y = w$ is *impossible* or *inconsistent*; sensors have failed or model is wrong

$y = Ax$ represents **output** resulting from input x

- v is a possible result or output
- w cannot be a result or output

$\mathcal{R}(A)$ characterizes the *possible results* or *achievable outputs*

Inverse

$A \in \mathbf{R}^{n \times n}$ is *invertible* or *nonsingular* if $\det A \neq 0$

equivalent conditions:

- columns of A are a basis for $\mathbf{R}^n$
- rows of A are a basis for $\mathbf{R}^n$
- $y = Ax$ has a unique solution x for every $y \in \mathbf{R}^n$
- A has a (left and right) inverse denoted $A^{-1} \in \mathbf{R}^{n \times n}$, with $AA^{-1} = A^{-1}A = I$
- $\mathcal{N}(A) = \{0\}$
- $\mathcal{R}(A) = \mathbf{R}^n$
- $\det A^T A = \det AA^T \neq 0$

Interpretations of inverse

suppose $A \in \mathbf{R}^{n \times n}$ has inverse $B = A^{-1}$

- mapping associated with B undoes mapping associated with A (applied either before or after!)
- $x = By$ is a perfect (pre- or post-) *equalizer* for the *channel* $y = Ax$
- $x = By$ is unique solution of $Ax = y$

Dual basis interpretation

- let a_i be columns of A, and $\tilde{b}_i^T$ be rows of $B = A^{-1}$
- from $y = x_1 a_1 + \cdots + x_n a_n$ and $x_i = \tilde{b}_i^T y$, we get

$$y = \sum_{i=1}^{n} (\tilde{b}_i^T y) a_i$$

  thus, inner product with *rows of inverse matrix* gives the coefficients in the *expansion of a vector in the columns of the matrix*

- $\tilde{b}_1, \ldots, \tilde{b}_n$ and $a_1, \ldots, a_n$ are called *dual bases*

Rank of a matrix

we define the *rank* of $A \in \mathbf{R}^{m \times n}$ as

$$\mathbf{rank}(A) = \mathbf{dim}\,\mathcal{R}(A)$$

(nontrivial) **facts:**

- $\mathbf{rank}(A) = \mathbf{rank}(A^T)$
- $\mathbf{rank}(A)$ is maximum number of independent columns (or rows) of A hence $\mathbf{rank}(A) \leq \mathbf{min}(m, n)$
- $\mathbf{rank}(A) + \mathbf{dim}\,\mathcal{N}(A) = n$

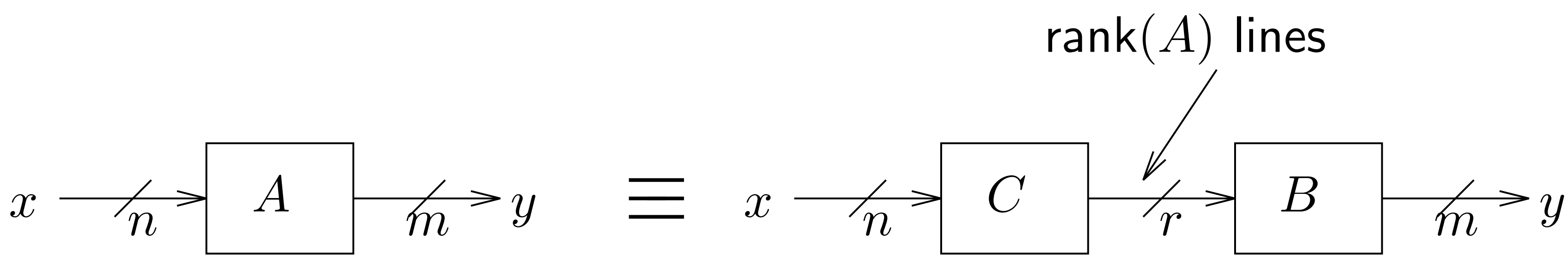
Conservation of dimension

interpretation of $\textbf{rank}(A) + \textbf{dim}\,\mathcal{N}(A) = n$:

- $\textbf{rank}(A)$ is dimension of set 'hit' by the mapping $y = Ax$
- $\textbf{dim}\,\mathcal{N}(A)$ is dimension of set of x 'crushed' to zero by $y = Ax$
- 'conservation of dimension': each dimension of input is either crushed to zero or ends up in output
- roughly speaking:
  - n is number of degrees of freedom in input x
  - $\textbf{dim}\,\mathcal{N}(A)$ is number of degrees of freedom lost in the mapping from x to $y = Ax$
  - $\textbf{rank}(A)$ is number of degrees of freedom in output y

‘Coding’ interpretation of rank

- rank of product: $\textbf{rank}(BC) \leq \textbf{min}\{\textbf{rank}(B),\ \textbf{rank}(C)\}$
- hence if $A = BC$ with $B \in \mathbf{R}^{m\times r}$, $C \in \mathbf{R}^{r\times n}$, then $\textbf{rank}(A) \leq r$
- conversely: if $\textbf{rank}(A) = r$ then $A \in \mathbf{R}^{m\times n}$ can be factored as $A = BC$ with $B \in \mathbf{R}^{m\times r}$, $C \in \mathbf{R}^{r\times n}$:



- $\textbf{rank}(A) = r$ is minimum size of vector needed to faithfully reconstruct y from x

Application: fast matrix-vector multiplication

- need to compute matrix-vector product $y = Ax$, $A \in \mathbf{R}^{m \times n}$
- A has known factorization $A = BC$, $B \in \mathbf{R}^{m \times r}$
- computing $y = Ax$ directly: mn operations
- computing $y = Ax$ as $y = B(Cx)$ (compute $z = Cx$ first, then $y = Bz$): $rn + mr = (m + n)r$ operations
- savings can be considerable if $r \ll \min\{m, n\}$

Full rank matrices

for $A \in \mathbf{R}^{m \times n}$ we always have $\textbf{rank}(A) \leq \textbf{min}(m, n)$

we say A is *full rank* if $\textbf{rank}(A) = \textbf{min}(m, n)$

- for **square** matrices, full rank means nonsingular
- for **skinny** matrices ($m \geq n$), full rank means columns are independent
- for **fat** matrices ($m \leq n$), full rank means rows are independent

Change of coordinates

'standard' basis vectors in $\mathbf{R}^n$: $(e_1, e_2, \ldots, e_n)$ where

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

(1 in ith component)

obviously we have

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

x_i are called the coordinates of x (in the standard basis) if $(t_1, t_2, \ldots, t_n)$ is another basis for $\mathbf{R}^n$, we have

$$x = \tilde{x}_1 t_1 + \tilde{x}_2 t_2 + \cdots + \tilde{x}_n t_n$$

where $\tilde{x}_i$ are the coordinates of x in the basis $(t_1, t_2, \ldots, t_n)$

define $T = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}$ so $x = T\tilde{x}$, hence

$$\tilde{x} = T^{-1}x$$

(T is invertible since t_i are a basis)

T^{-1} transforms (standard basis) coordinates of x into t_i-coordinates

inner product ith row of T^{-1} with x extracts t_i-coordinate of x

consider linear transformation $y = Ax$, $A \in \mathbf{R}^{n\times n}$

express y and x in terms of $t_1, t_2 \ldots, t_n$:

$$x = T\tilde{x}, \quad y = T\tilde{y}$$

so

$$\tilde{y} = (T^{-1}AT)\tilde{x}$$

- $A \longrightarrow T^{-1}AT$ is called *similarity transformation*
- similarity transformation by T expresses linear transformation $y = Ax$ in coordinates $t_1, t_2, \ldots, t_n$

(Euclidean) norm

for $x \in \mathbf{R}^n$ we define the (Euclidean) norm as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

$\|x\|$ measures length of vector (from origin)

important properties:

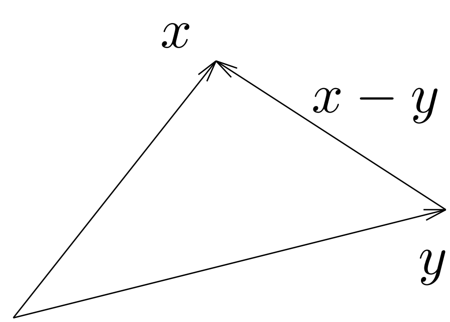
- $\|\alpha x\| = |\alpha|\|x\|$ (homogeneity)
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)
- $\|x\| \geq 0$ (nonnegativity)
- $\|x\| = 0 \iff x = 0$ (definiteness)

RMS value and (Euclidean) distance

root-mean-square (RMS) value of vector $x \in \mathbf{R}^n$:

$$\mathbf{rms}(x) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^2 \right)^{1/2} = \frac{\|x\|}{\sqrt{n}}$$

norm defines distance between vectors: $\mathbf{dist}(x, y) = \|x - y\|$



Inner product

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^T y$$

important properties:

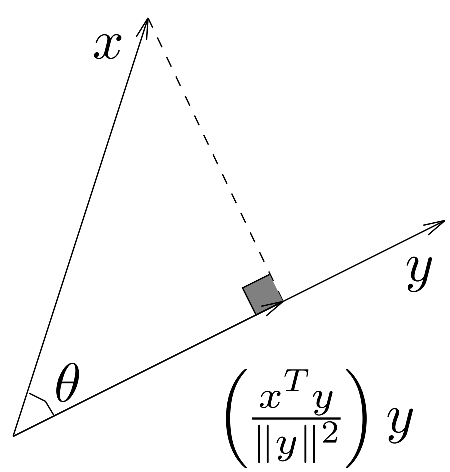
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, x \rangle \geq 0$
- $\langle x, x \rangle = 0 \iff x = 0$

$f(y) = \langle x, y \rangle$ is linear function : $\mathbf{R}^n \to \mathbf{R}$, with linear map defined by row vector x^T

Cauchy-Schwartz inequality and angle between vectors

- for any $x, y \in \mathbf{R}^n$, $|x^T y| \leq \|x\|\|y\|$
- (unsigned) angle between vectors in $\mathbf{R}^n$ defined as

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\|\|y\|}$$



thus $x^T y = \|x\|\|y\| \cos \theta$

special cases:

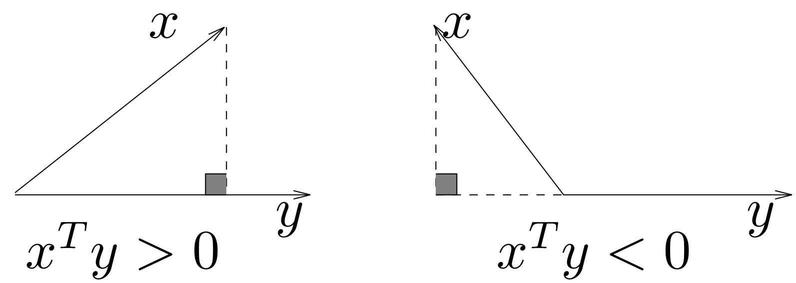
- x and y are *aligned*: $\theta = 0$; $x^T y = \|x\|\|y\|$;
  (if $x \neq 0$) $y = \alpha x$ for some $\alpha \geq 0$

- x and y are *opposed*: $\theta = \pi$; $x^T y = -\|x\|\|y\|$
  (if $x \neq 0$) $y = -\alpha x$ for some $\alpha \geq 0$

- x and y are *orthogonal*: $\theta = \pi/2$ or $-\pi/2$; $x^T y = 0$
  denoted $x \perp y$

interpretation of $x^T y > 0$ and $x^T y < 0$:

- $x^T y > 0$ means $\angle(x, y)$ is acute
- $x^T y < 0$ means $\angle(x, y)$ is obtuse



$\{x \mid x^T y \leq 0\}$ defines a *halfspace* with outward normal vector y, and boundary passing through 0