

# Lecture 13

## Linear dynamical systems with inputs & outputs

- inputs & outputs: interpretations
- transfer matrix
- impulse and step matrices
- examples

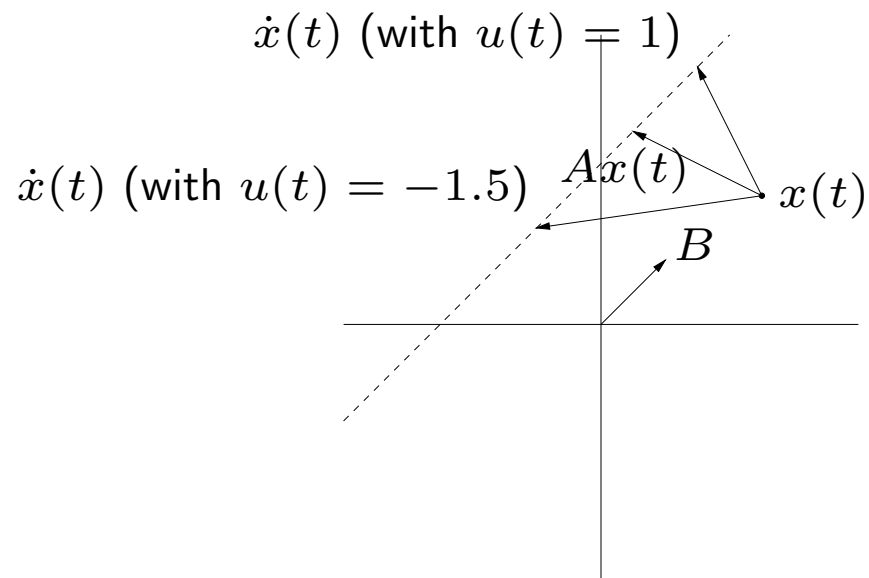
# Inputs & outputs

recall continuous-time time-invariant LDS has form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

- $Ax$  is called the *drift term* (of  $\dot{x}$ )
- $Bu$  is called the *input term* (of  $\dot{x}$ )

picture, with  $B \in \mathbf{R}^{2 \times 1}$ :



# Interpretations

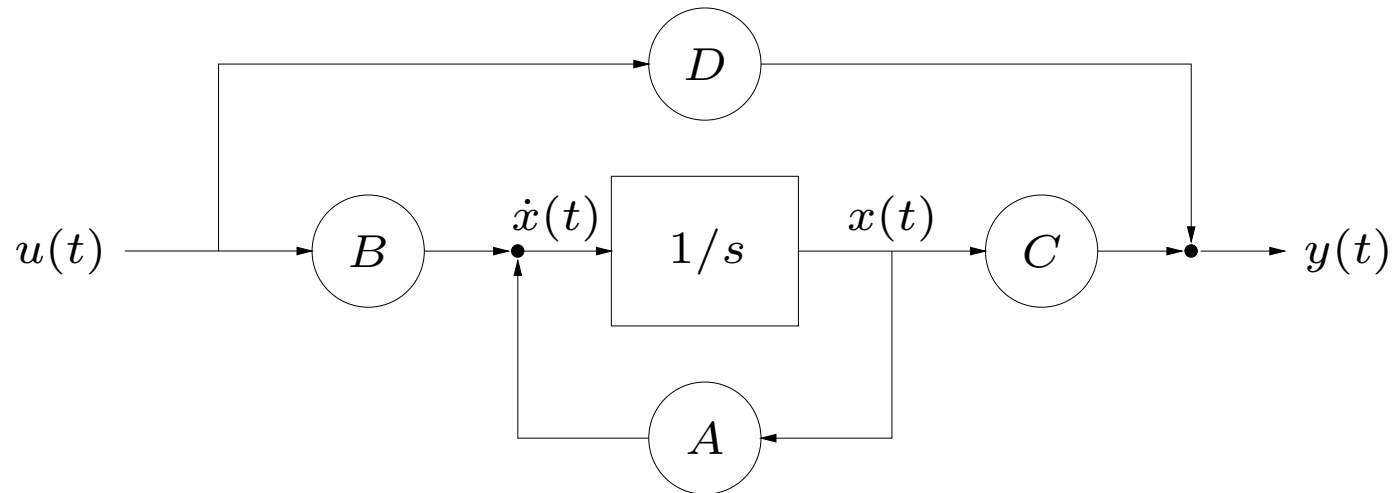
write  $\dot{x} = Ax + b_1u_1 + \cdots + b_mu_m$ , where  $B = [b_1 \cdots b_m]$

- state derivative is sum of autonomous term ( $Ax$ ) and one term per input ( $b_iu_i$ )
- each input  $u_i$  gives another degree of freedom for  $\dot{x}$  (assuming columns of  $B$  independent)

write  $\dot{x} = Ax + Bu$  as  $\dot{x}_i = \tilde{a}_i^T x + \tilde{b}_i^T u$ , where  $\tilde{a}_i^T, \tilde{b}_i^T$  are the rows of  $A, B$

- $i$ th state derivative is linear function of state  $x$  and input  $u$

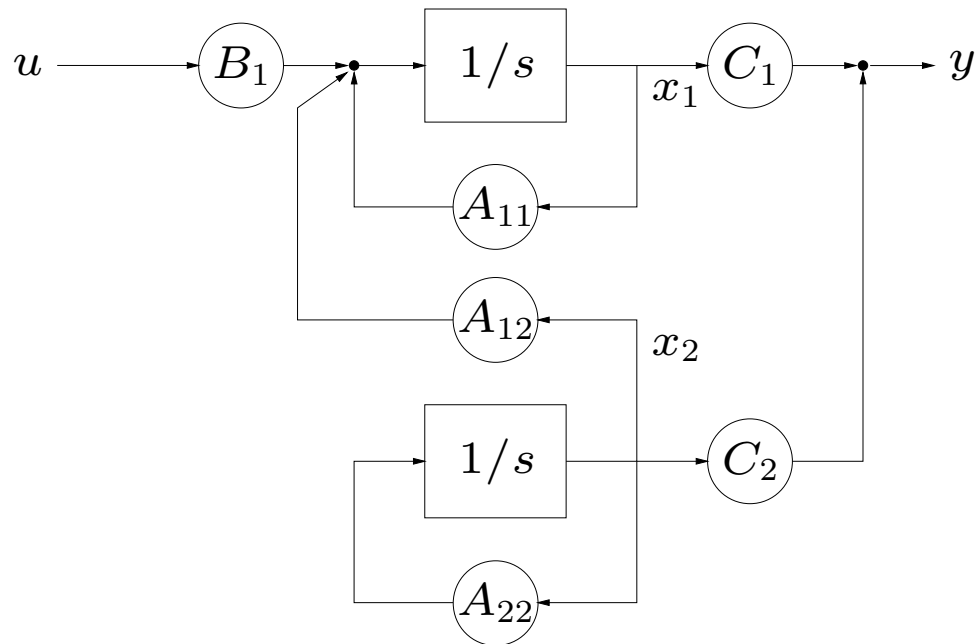
## Block diagram



- $A_{ij}$  is gain factor from state  $x_j$  into integrator  $i$
- $B_{ij}$  is gain factor from input  $u_j$  into integrator  $i$
- $C_{ij}$  is gain factor from state  $x_j$  into output  $y_i$
- $D_{ij}$  is gain factor from input  $u_j$  into output  $y_i$

interesting when there is structure, *e.g.*, with  $x_1 \in \mathbf{R}^{n_1}$ ,  $x_2 \in \mathbf{R}^{n_2}$ :

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



- $x_2$  is not affected by input  $u$ , *i.e.*,  $x_2$  propagates autonomously
- $x_2$  affects  $y$  directly and through  $x_1$

# Transfer matrix

take Laplace transform of  $\dot{x} = Ax + Bu$ :

$$sX(s) - x(0) = AX(s) + BU(s)$$

hence

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

so

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) d\tau$$

- $e^{tA}x(0)$  is the unforced or autonomous response
- $e^{tA}B$  is called the input-to-state impulse matrix
- $(sI - A)^{-1}B$  is called the *input-to-state transfer matrix* or *transfer function*

with  $y = Cx + Du$  we have:

$$Y(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)U(s)$$

so

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) d\tau + Du(t)$$

- output term  $Ce^{tA}x(0)$  due to initial condition
- $H(s) = C(sI - A)^{-1}B + D$  is called the *transfer function* or *transfer matrix*
- $h(t) = Ce^{tA}B + D\delta(t)$  is called the *impulse matrix* or *impulse response* ( $\delta$  is the Dirac delta function)

with zero initial condition we have:

$$Y(s) = H(s)U(s), \quad y = h * u$$

where  $*$  is convolution (of matrix valued functions)

intepretation:

- $H_{ij}$  is transfer function from input  $u_j$  to output  $y_i$



# Impulse matrix

impulse matrix  $h(t) = Ce^{tA}B + D\delta(t)$

with  $x(0) = 0$ ,  $y = h * u$ , *i.e.*,

$$y_i(t) = \sum_{j=1}^m \int_0^t h_{ij}(t - \tau) u_j(\tau) d\tau$$

## interpretations:

- $h_{ij}(t)$  is impulse response from  $j$ th input to  $i$ th output
- $h_{ij}(t)$  gives  $y_i$  when  $u(t) = e_j\delta$
- $h_{ij}(\tau)$  shows how dependent output  $i$  is, on what input  $j$  was,  $\tau$  seconds ago
- $i$  indexes output;  $j$  indexes input;  $\tau$  indexes time lag

# Step matrix

the *step matrix* or *step response matrix* is given by

$$s(t) = \int_0^t h(\tau) d\tau$$

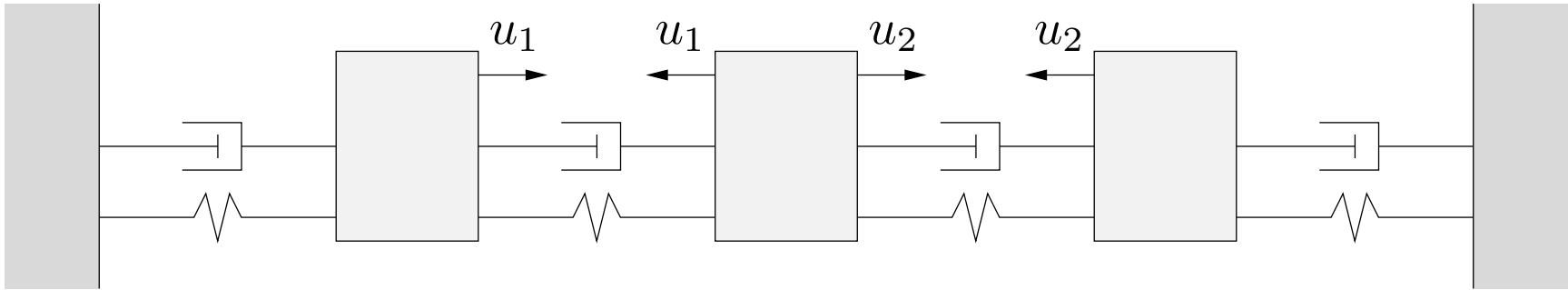
**interpretations:**

- $s_{ij}(t)$  is step response from  $j$ th input to  $i$ th output
- $s_{ij}(t)$  gives  $y_i$  when  $u = e_j$  for  $t \geq 0$

for invertible  $A$ , we have

$$s(t) = CA^{-1} (e^{tA} - I) B + D$$

# Example 1



- unit masses, springs, dampers
- $u_1$  is tension between 1st & 2nd masses
- $u_2$  is tension between 2nd & 3rd masses
- $y \in \mathbf{R}^3$  is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

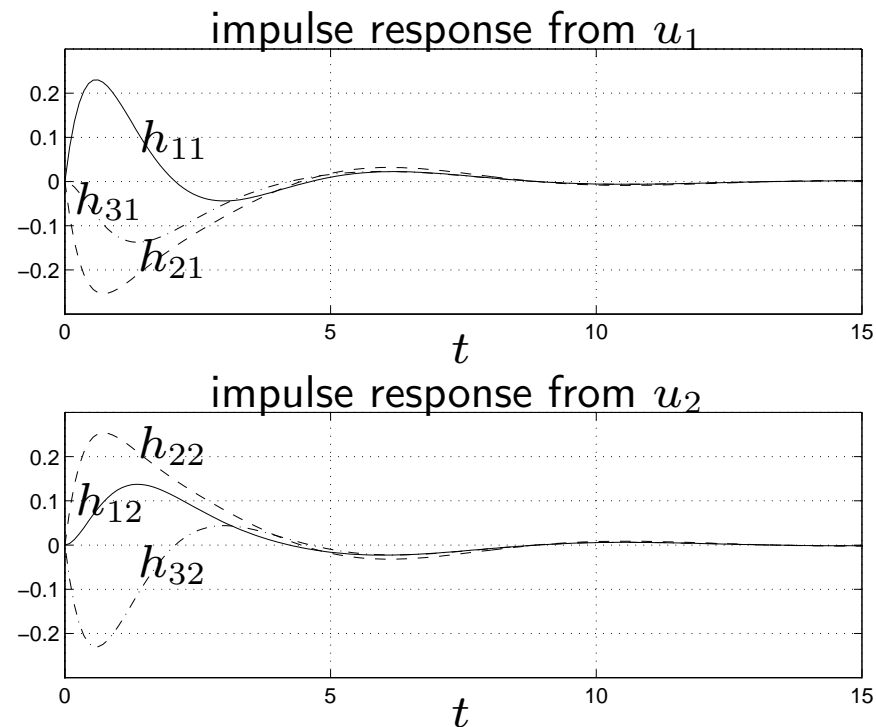
system is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

eigenvalues of  $A$  are

$$-1.71 \pm j0.71, \quad -1.00 \pm j1.00, \quad -0.29 \pm j0.71$$

impulse matrix:

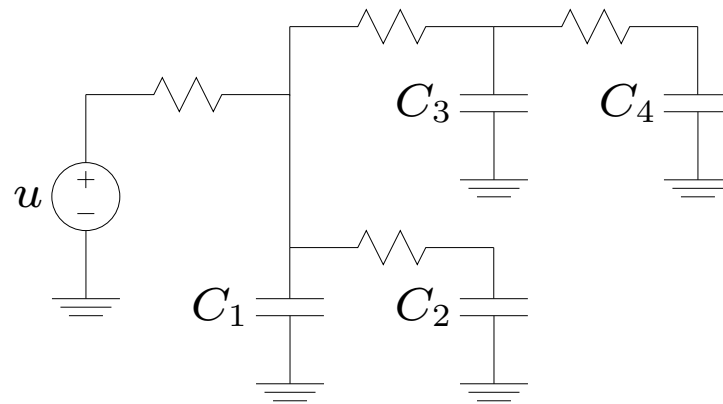


roughly speaking:

- impulse at  $u_1$  affects third mass less than other two
- impulse at  $u_2$  affects first mass later than other two

## Example 2

interconnect circuit:



- $u(t) \in \mathbf{R}$  is input (drive) voltage
- $x_i$  is voltage across  $C_i$
- output is state:  $y = x$
- unit resistors, unit capacitors
- step response matrix shows delay to each node

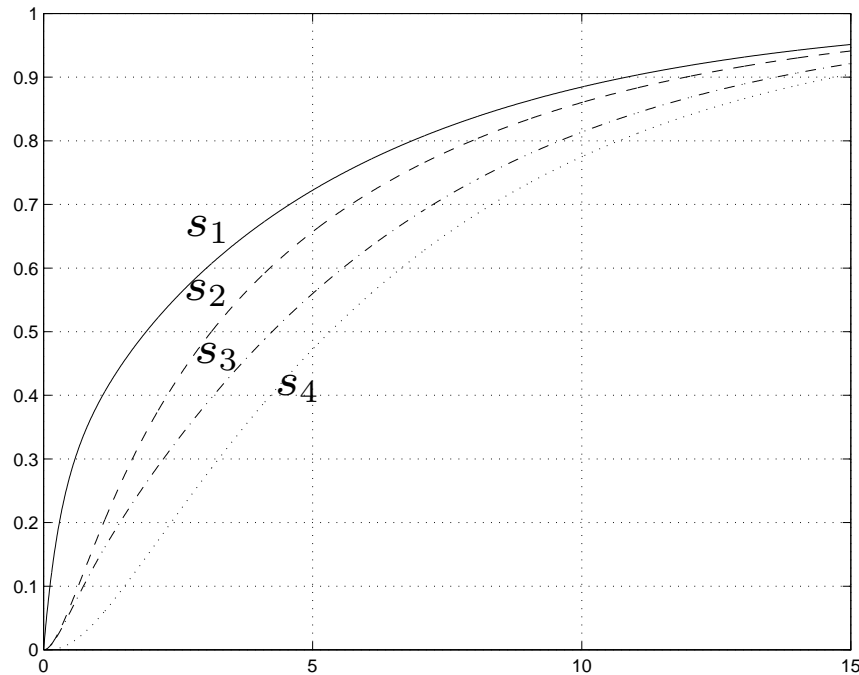
system is

$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \quad y = x$$

eigenvalues of  $A$  are

$$-0.17, \quad -0.66, \quad -2.21, \quad -3.96$$

step response matrix  $s(t) \in \mathbf{R}^{4 \times 1}$ :



- shortest delay to  $x_1$ ; longest delay to  $x_4$
- delays  $\approx 10$ , consistent with slowest (*i.e.*, dominant) eigenvalue  $-0.17$



## DC or static gain matrix

- transfer matrix at  $s = 0$  is  $H(0) = -CA^{-1}B + D \in \mathbf{R}^{m \times p}$
- DC transfer matrix describes system under *static* conditions, *i.e.*,  $x$ ,  $u$ ,  $y$  constant:

$$0 = \dot{x} = Ax + Bu, \quad y = Cx + Du$$

eliminate  $x$  to get  $y = H(0)u$

- if system is stable,

$$H(0) = \int_0^{\infty} h(t) dt = \lim_{t \rightarrow \infty} s(t)$$

$$\text{(recall: } H(s) = \int_0^{\infty} e^{-st} h(t) dt, \quad s(t) = \int_0^t h(\tau) d\tau)$$

if  $u(t) \rightarrow u_{\infty} \in \mathbf{R}^m$ , then  $y(t) \rightarrow y_{\infty} \in \mathbf{R}^p$  where  $y_{\infty} = H(0)u_{\infty}$

DC gain matrix for example 1 (springs):

$$H(0) = \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{bmatrix}$$

DC gain matrix for example 2 (RC circuit):

$$H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(do these make sense?)

# Discretization with piecewise constant inputs

linear system  $\dot{x} = Ax + Bu, y = Cx + Du$

suppose  $u_d : \mathbf{Z}_+ \rightarrow \mathbf{R}^m$  is a sequence, and

$$u(t) = u_d(k) \quad \text{for } kh \leq t < (k+1)h, \quad k = 0, 1, \dots$$

define sequences

$$x_d(k) = x(kh), \quad y_d(k) = y(kh), \quad k = 0, 1, \dots$$

- $h > 0$  is called the *sample interval* (for  $x$  and  $y$ ) or *update interval* (for  $u$ )
- $u$  is piecewise constant (called *zero-order-hold*)
- $x_d, y_d$  are sampled versions of  $x, y$

$$\begin{aligned}
x_d(k+1) &= x((k+1)h) \\
&= e^{hA}x(kh) + \int_0^h e^{\tau A} B u((k+1)h - \tau) d\tau \\
&= e^{hA}x_d(k) + \left( \int_0^h e^{\tau A} d\tau \right) B u_d(k)
\end{aligned}$$

$x_d$ ,  $u_d$ , and  $y_d$  satisfy discrete-time LDS equations

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k), \quad y_d(k) = C_d x_d(k) + D_d u_d(k)$$

where

$$A_d = e^{hA}, \quad B_d = \left( \int_0^h e^{\tau A} d\tau \right) B, \quad C_d = C, \quad D_d = D$$

called *discretized system*

if  $A$  is invertible, we can express integral as

$$\int_0^h e^{\tau A} d\tau = A^{-1} (e^{hA} - I)$$

**stability:** if eigenvalues of  $A$  are  $\lambda_1, \dots, \lambda_n$ , then eigenvalues of  $A_d$  are  $e^{h\lambda_1}, \dots, e^{h\lambda_n}$

discretization preserves stability properties since

$$\Re \lambda_i < 0 \quad \Leftrightarrow \quad |e^{h\lambda_i}| < 1$$

for  $h > 0$

## extensions/variations:

- *offsets*: updates for  $u$  and sampling of  $x, y$  are offset in time
- *multirate*:  $u_i$  updated,  $y_i$  sampled at different intervals  
(usually integer multiples of a common interval  $h$ )

both very common in practice

# Dual system

the *dual system* associated with system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is given by

$$\dot{z} = A^T z + C^T v, \quad w = B^T z + D^T v$$

- all matrices are transposed
- role of  $B$  and  $C$  are swapped

transfer function of dual system:

$$(B^T)(sI - A^T)^{-1}(C^T) + D^T = H(s)^T$$

where  $H(s) = C(sI - A)^{-1}B + D$

(for SISO case, TF of dual is same as original)  
eigenvalues (hence stability properties) are the same

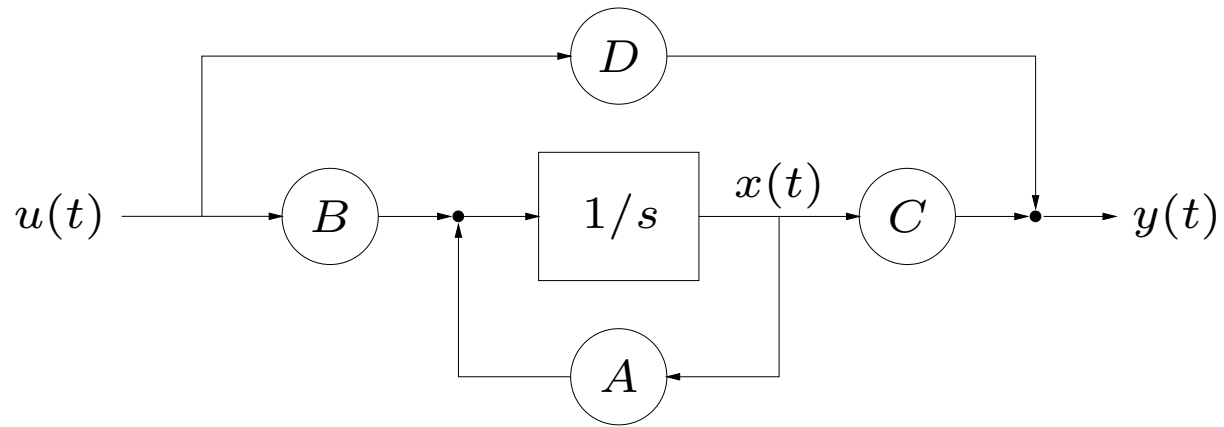


# Dual via block diagram

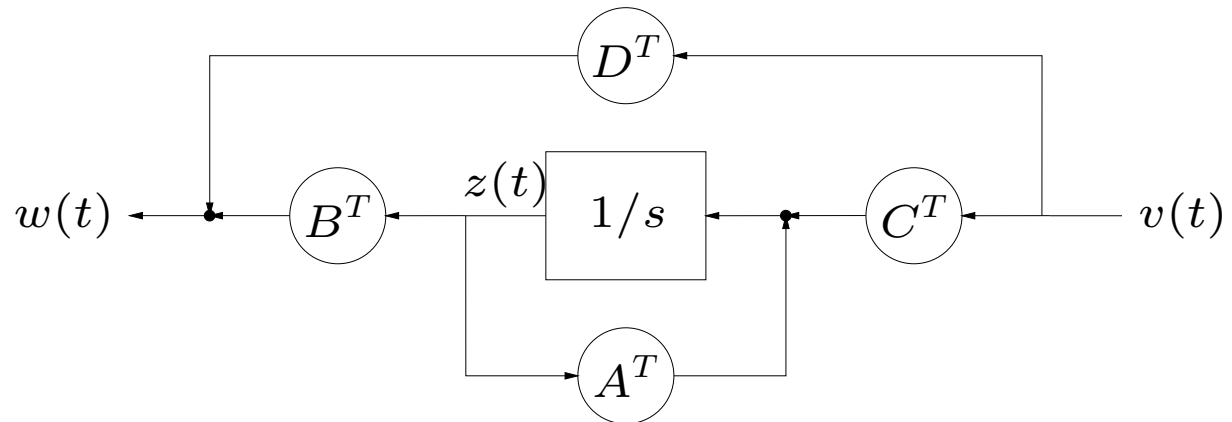
in terms of block diagrams, dual is formed by:

- transpose all matrices
- swap inputs and outputs on all boxes
- reverse directions of signal flow arrows
- swap solder joints and summing junctions

original system:



dual system:



# Causality

interpretation of

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) d\tau$$

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) d\tau + Du(t)$$

for  $t \geq 0$ :

*current* state ( $x(t)$ ) and output ( $y(t)$ ) depend on *past* input ( $u(\tau)$  for  $\tau \leq t$ )

*i.e.*, mapping from input to state and output is *causal* (with fixed *initial* state)

now consider fixed *final* state  $x(T)$ : for  $t \leq T$ ,

$$x(t) = e^{(t-T)A}x(T) + \int_T^t e^{(t-\tau)A}Bu(\tau) d\tau,$$

*i.e.*, current state (and output) depend on future input!

so for fixed final condition, same system is anti-causal

# Idea of state

$x(t)$  is called *state* of system at time  $t$  since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs

## Change of coordinates

start with LDS  $\dot{x} = Ax + Bu$ ,  $y = Cx + Du$

change coordinates in  $\mathbf{R}^n$  to  $\tilde{x}$ , with  $x = T\tilde{x}$

then

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

hence LDS can be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \quad y = \tilde{C}\tilde{x} + \tilde{D}u$$

where

$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = CT, \quad \tilde{D} = D$$

TF is same (since  $u$ ,  $y$  aren't affected):

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI - A)^{-1}B + D$$

# Standard forms for LDS

can change coordinates to put  $A$  in various forms (diagonal, real modal, Jordan . . . )

*e.g.*, to put LDS in *diagonal form*, find  $T$  s.t.

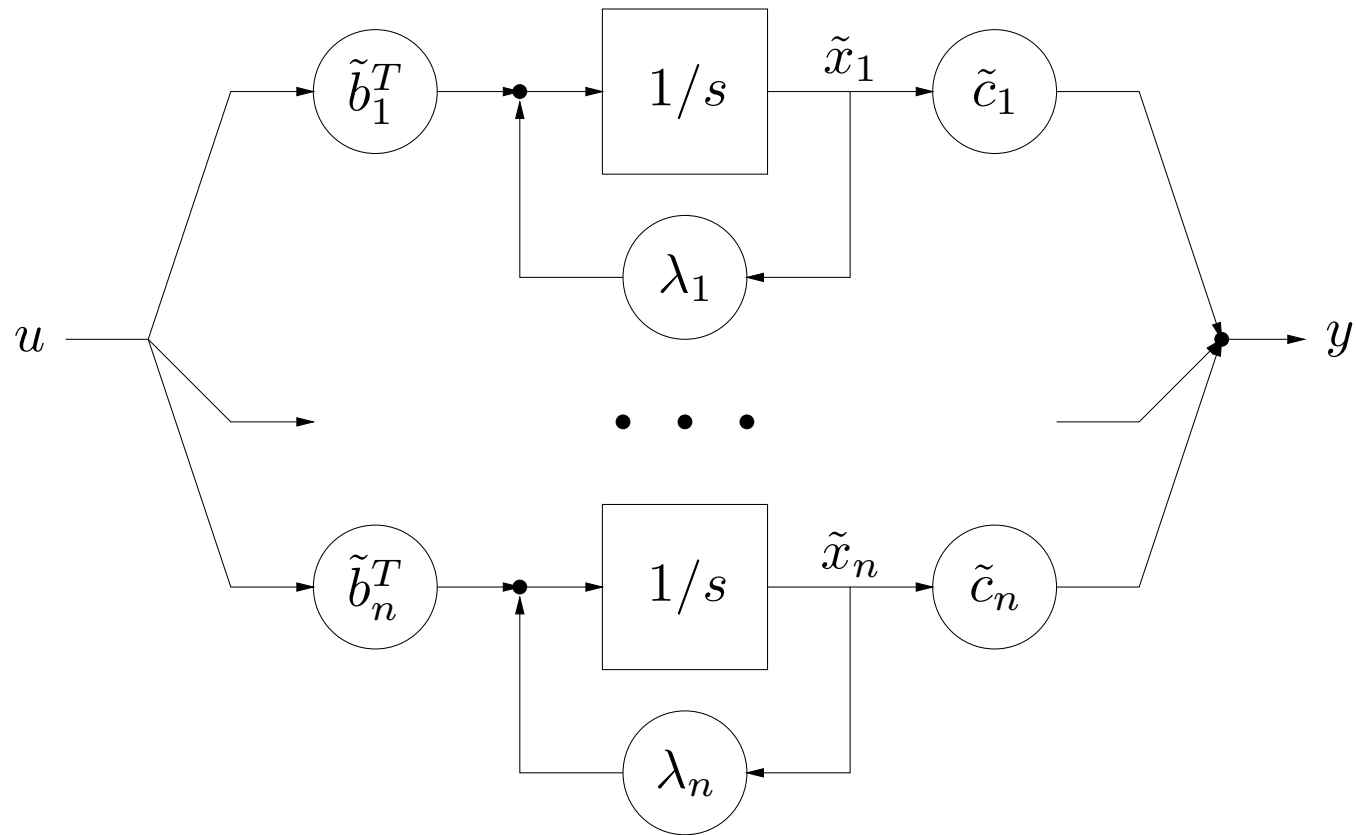
$$T^{-1}AT = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^T \\ \vdots \\ \tilde{b}_n^T \end{bmatrix}, \quad CT = [ \tilde{c}_1 \quad \cdots \quad \tilde{c}_n ]$$

so

$$\dot{\tilde{x}}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^T u, \quad y = \sum_{i=1}^n \tilde{c}_i \tilde{x}_i$$



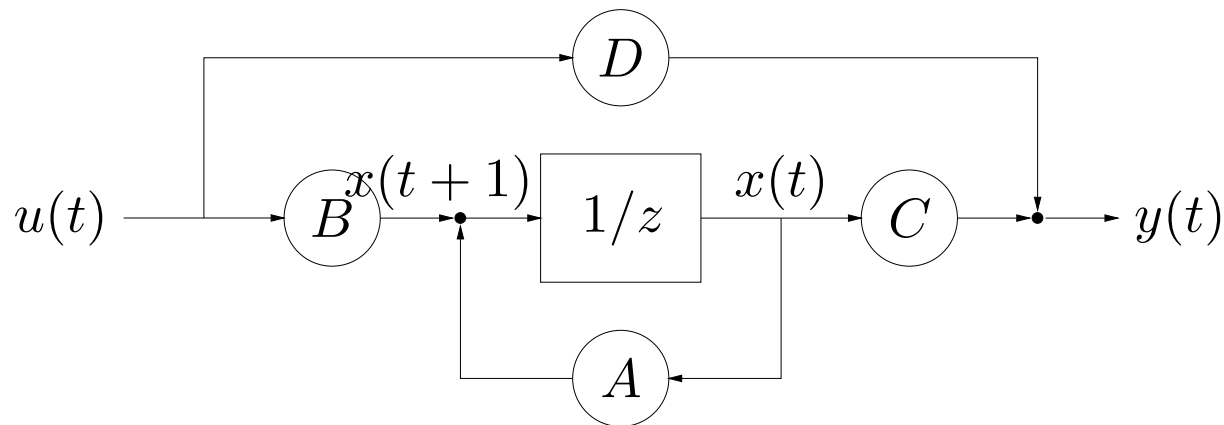
(here we assume  $D = 0$ )



# Discrete-time systems

discrete-time LDS:

$$x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$



- only difference w/cts-time:  $z$  instead of  $s$
- interpretation of  $z^{-1}$  block:
  - unit delayor (shifts sequence back in time one epoch)
  - latch (plus small delay to avoid race condition)

we have:

$$x(1) = Ax(0) + Bu(0),$$

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) \\ &= A^2x(0) + ABu(0) + Bu(1), \end{aligned}$$

and in general, for  $t \in \mathbf{Z}_+$ ,

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau)$$

hence

$$y(t) = CA^t x(0) + h * u$$

where  $*$  is discrete-time convolution and

$$h(t) = \begin{cases} D, & t = 0 \\ CA^{t-1}B, & t > 0 \end{cases}$$

is the impulse response

## $\mathcal{Z}$ -transform

suppose  $w \in \mathbf{R}^{p \times q}$  is a sequence (discrete-time signal), *i.e.*,

$$w : \mathbf{Z}_+ \rightarrow \mathbf{R}^{p \times q}$$

recall  $\mathcal{Z}$ -transform  $W = \mathcal{Z}(w)$ :

$$W(z) = \sum_{t=0}^{\infty} z^{-t} w(t)$$

where  $W : D \subseteq \mathbf{C} \rightarrow \mathbf{C}^{p \times q}$  ( $D$  is domain of  $W$ )

time-advanced or shifted signal  $v$ :

$$v(t) = w(t + 1), \quad t = 0, 1, \dots$$

$\mathcal{Z}$ -transform of time-advanced signal:

$$\begin{aligned} V(z) &= \sum_{t=0}^{\infty} z^{-t} w(t+1) \\ &= z \sum_{t=1}^{\infty} z^{-t} w(t) \\ &= zW(z) - zw(0) \end{aligned}$$

# Discrete-time transfer function

take  $\mathcal{Z}$ -transform of system equations

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

yields

$$zX(z) - zx(0) = AX(z) + BU(z), \quad Y(z) = CX(z) + DU(z)$$

solve for  $X(z)$  to get

$$X(z) = (zI - A)^{-1}zx(0) + (zI - A)^{-1}BU(z)$$

(note extra  $z$  in first term!)

hence

$$Y(z) = H(z)U(z) + C(zI - A)^{-1}zx(0)$$

where  $H(z) = C(zI - A)^{-1}B + D$  is the *discrete-time transfer function*

note power series expansion of resolvent:

$$(zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \dots$$