Lecture 18
Controllability and state transfer

• state transfer

• reachable set, controllability matrix

• minimum norm inputs

• infinite-horizon minimum norm transfer
State transfer

consider \( \dot{x} = Ax + Bu \) (or \( x(t+1) = Ax(t) + Bu(t) \)) over time interval \([t_i, t_f]\)

we say input \( u : [t_i, t_f] \rightarrow \mathbb{R}^m \) steers or transfers state from \( x(t_i) \) to \( x(t_f) \)
(over time interval \([t_i, t_f]\))

(subscripts stand for initial and final)

questions:

- where can \( x(t_i) \) be transfered to at \( t = t_f \)?
- how quickly can \( x(t_i) \) be transfered to some \( x_{\text{target}} \)?
- how do we find a \( u \) that transfers \( x(t_i) \) to \( x(t_f) \)?
- how do we find a ‘small’ or ‘efficient’ \( u \) that transfers \( x(t_i) \) to \( x(t_f) \)?
Reachability

consider state transfer from \( x(0) = 0 \) to \( x(t) \)
we say \( x(t) \) is reachable (in \( t \) seconds or epochs)
we define \( R_t \subseteq \mathbb{R}^n \) as the set of points reachable in \( t \) seconds or epochs
for CT system \( \dot{x} = Ax + Bu \),

\[
R_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau) \, d\tau \left| u : [0, t] \rightarrow \mathbb{R}^m \right. \right\}
\]

and for DT system \( x(t + 1) = Ax(t) + Bu(t) \),

\[
R_t = \left\{ \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) \left| u(t) \in \mathbb{R}^m \right. \right\}
\]
• $\mathcal{R}_t$ is a subspace of $\mathbb{R}^n$

• $\mathcal{R}_t \subseteq \mathcal{R}_s$ if $t \leq s$

  (i.e., can reach more points given more time)

we define the reachable set $\mathcal{R}$ as the set of points reachable for some $t$:

$$\mathcal{R} = \bigcup_{t \geq 0} \mathcal{R}_t$$
Reachability for discrete-time LDS

DT system \( x(t + 1) = Ax(t) + Bu(t), \ x(t) \in \mathbb{R}^n \)

\[ x(t) = C_t \begin{bmatrix} u(t - 1) \\ \vdots \\ u(0) \end{bmatrix} \]

where \( C_t = \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix} \)

so reachable set at \( t \) is \( R_t = \text{range}(C_t) \)

by C-H theorem, we can express each \( A^k \) for \( k \geq n \) as linear combination of \( A^0, \ldots, A^{n-1} \)

hence for \( t \geq n \), \( \text{range}(C_t) = \text{range}(C_n) \)
thus we have

$$\mathcal{R}_t = \begin{cases} \text{range}(C_t) & t < n \\ \text{range}(C) & t \geq n \end{cases}$$

where $C = C_n$ is called the \textit{controllability matrix}

- any state that can be reached can be reached by $t = n$
- the reachable set is $\mathcal{R} = \text{range}(C)$
Controllable system

system is called *reachable* or *controllable* if all states are reachable (i.e., $\mathcal{R} = \mathbb{R}^n$)

system is reachable if and only if $\text{Rank}(C) = n$

**example:** $x(t + 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$

controllability matrix is $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

hence system is not controllable; reachable set is

$$\mathcal{R} = \text{range}(C) = \{ x \mid x_1 = x_2 \}$$
General state transfer

with \( t_f > t_i \),

\[
x(t_f) = A^{t_f-t_i}x(t_i) + C_{t_f-t_i}\begin{bmatrix} u(t_f-1) \\ \vdots \\ u(t_i) \end{bmatrix}
\]

hence can transfer \( x(t_i) \) to \( x(t_f) = x_{\text{des}} \)

\[
\Leftrightarrow \quad x_{\text{des}} - A^{t_f-t_i}x(t_i) \in \mathcal{R}_{t_f-t_i}
\]

• general state transfer reduces to reachability problem
• if system is controllable any state transfer can be achieved in \( \leq n \) steps
• important special case: driving state to zero (sometimes called regulating or controlling state)
Least-norm input for reachability

assume system is reachable, \( \text{Rank}(C_t) = n \)

to steer \( x(0) = 0 \) to \( x(t) = x_{\text{des}} \), inputs \( u(0), \ldots, u(t-1) \) must satisfy

\[
x_{\text{des}} = C_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}
\]

among all \( u \) that steer \( x(0) = 0 \) to \( x(t) = x_{\text{des}} \), the one that minimizes

\[
\sum_{\tau=0}^{t-1} \|u(\tau)\|^2
\]
is given by
\[
\begin{bmatrix}
  u_{\text{ln}}(t - 1) \\
  \vdots \\
  u_{\text{ln}}(0)
\end{bmatrix}
= C_t^T (C_t C_t^T)^{-1} x_{\text{des}}
\]

\(u_{\text{ln}}\) is called least-norm or minimum energy input that effects state transfer.

can express as
\[
u_{\text{ln}}(\tau) = B^T (A^T)^{(t-1-\tau)} \left( \sum_{s=0}^{t-1} A^s B B^T (A^T)^s \right)^{-1} x_{\text{des}},
\]
for \(\tau = 0, \ldots, t - 1\)
\( E_{\text{min}} \), the minimum value of \( \sum_{\tau=0}^{t-1} \| u(\tau) \|^2 \) required to reach \( x(t) = x_{\text{des}} \), is sometimes called \textit{minimum energy} required to reach \( x(t) = x_{\text{des}} \)

\[
E_{\text{min}} = \sum_{\tau=0}^{t-1} \| u_{\ln}(\tau) \|^2 \\
= \left( C_t^T (C_t C_t^T)^{-1} x_{\text{des}} \right)^T C_t^T (C_t C_t^T)^{-1} x_{\text{des}} \\
= x_{\text{des}}^T (C_t C_t^T)^{-1} x_{\text{des}} \\
= x_{\text{des}}^T \left( \sum_{\tau=0}^{t-1} A^T B B^T (A^T)^\tau \right)^{-1} x_{\text{des}}
\]
• $\mathcal{E}_{\text{min}}(x_{\text{des}}, t)$ gives measure of how hard it is to reach $x(t) = x_{\text{des}}$ from $x(0) = 0$ (i.e., how large a $u$ is required)

• $\mathcal{E}_{\text{min}}(x_{\text{des}}, t)$ gives practical measure of controllability/reachability (as function of $x_{\text{des}}, t$)

• ellipsoid $\{ z | \mathcal{E}_{\text{min}}(z, t) \leq 1 \}$ shows points in state space reachable at $t$ with one unit of energy

(shows directions that can be reached with small inputs, and directions that can be reached only with large inputs)
$\mathcal{E}_{\text{min}}$ as function of $t$:

if $t \geq s$ then

$$\sum_{\tau=0}^{t-1} A^\tau B B^T (A^T)^\tau \geq \sum_{\tau=0}^{s-1} A^\tau B B^T (A^T)^\tau$$

hence

$$\left( \sum_{\tau=0}^{t-1} A^\tau B B^T (A^T)^\tau \right)^{-1} \leq \left( \sum_{\tau=0}^{s-1} A^\tau B B^T (A^T)^\tau \right)^{-1}$$

so $\mathcal{E}_{\text{min}}(x_{\text{des}}, t) \leq \mathcal{E}_{\text{min}}(x_{\text{des}}, s)$

i.e.: takes less energy to get somewhere more leisurely
example: \( x(t + 1) = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \)

\( \mathcal{E}_{\text{min}}(z, t) \) for \( z = [1 \ 1]^T \):

\[ \begin{array}{llllllllll} 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 \\ 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \]
ellipsoids $\mathcal{E}_{\min} \leq 1$ for $t = 3$ and $t = 10$: 

\begin{align*}
\text{when } t = 3 \quad &\text{and } t = 10:
\end{align*}
Minimum energy over infinite horizon

the matrix

\[
P = \lim_{t \to \infty} \left( \sum_{\tau=0}^{t-1} A^\tau B B^T (A^T)^\tau \right)^{-1}
\]

always exists, and gives the minimum energy required to reach a point \( x_{\text{des}} \) (with no limit on \( t \)):

\[
\min \left\{ \sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \right. \left| x(0) = 0, x(t) = x_{\text{des}} \right\} = x_{\text{des}}^T P x_{\text{des}}
\]

if \( A \) is stable, \( P > 0 \) (i.e., can’t get anywhere for free)

if \( A \) is not stable, then \( P \) can have nonzero nullspace
• $Pz = 0, z \neq 0$ means can get to $z$ using $u$’s with energy as small as you like

(u just gives a little kick to the state; the instability carries it out to $z$ efficiently)

• basis of highly maneuverable, unstable aircraft
Continuous-time reachability

consider now $\dot{x} = Ax + Bu$ with $x(t) \in \mathbb{R}^n$

reachable set at time $t$ is

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau) \, d\tau \middle| u : [0, t] \rightarrow \mathbb{R}^m \right\}$$

fact: for $t > 0$, $\mathcal{R}_t = \mathcal{R} = \text{range}(C)$, where

$$C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is the controllability matrix of $(A, B)$

- same $\mathcal{R}$ as discrete-time system

- for continuous-time system, any reachable point can be reached as fast as you like (with large enough $u$)
first let’s show for any \( u \) (and \( x(0) = 0 \)) we have \( x(t) \in \text{range}(C) \)

write \( e^{tA} \) as power series:

\[
e^{tA} = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \cdots
\]

by C-H, express \( A^n, A^{n+1}, \ldots \) in terms of \( A^0, \ldots, A^{n-1} \) and collect powers of \( A \):

\[
e^{tA} = \alpha_0(t)I + \alpha_1(t)A + \cdots + \alpha_{n-1}(t)A^{n-1}
\]

therefore

\[
x(t) = \int_0^t e^{\tau A} Bu(t - \tau) \, d\tau = \int_0^t \left( \sum_{i=0}^{n-1} \alpha_i(\tau) A^i \right) Bu(t - \tau) \, d\tau
\]
\[
\begin{align*}
&= \sum_{i=0}^{n-1} A^i B \int_0^t \alpha_i(\tau) u(t - \tau) \, d\tau \\
&= C z
\end{align*}
\]

where \( z_i = \int_0^t \alpha_i(\tau) u(t - \tau) \, d\tau \)

hence, \( x(t) \) is always in \( \text{range}(C) \)

need to show converse: every point in \( \text{range}(C) \) can be reached
Impulsive inputs

suppose $x(0_-) = 0$ and we apply input $u(t) = \delta^{(k)}(t)f$, where $\delta^{(k)}$ denotes $k$th derivative of $\delta$ and $f \in \mathbb{R}^m$

then $U(s) = s^k f$, so

$$X(s) = (sI - A)^{-1}Bs^k f$$
$$= (s^{-1}I + s^{-2}A + \cdots) Bs^k f$$
$$= (s^{k-1} + \cdots + sA^{k-2} + A^{k-1} + s^{-1}A^k + \cdots)Bf$$

impulsive terms

hence

$$x(t) = \text{impulsive terms} + A^kBf + A^{k+1}Bf \frac{t}{1!} + A^{k+2}Bf \frac{t^2}{2!} + \cdots$$

in particular, $x(0_+) = A^kBf$
thus, input \( u = \delta^{(k)} f \) transfers state from \( x(0-) = 0 \) to \( x(0+) = A^k B f \)

now consider input of form

\[
u(t) = \delta(t) f_0 + \cdots + \delta^{(n-1)}(t) f_{n-1}
\]

where \( f_i \in \mathbb{R}^m \)

by linearity we have

\[
x(0+) = B f_0 + \cdots + A^{n-1} B f_{n-1} = C \begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix}
\]

hence we can reach any point in \( \text{range}(C) \)

(at least, using impulse inputs)
can also be shown that any point in \( \text{range}(\mathcal{C}) \) can be reached for any \( t > 0 \) using nonimpulsive inputs

**Fact:** if \( x(0) \in \mathcal{R} \), then \( x(t) \in \mathcal{R} \) for all \( t \) (no matter what \( u \) is)

to show this, need to show \( e^{tA}x(0) \in \mathcal{R} \) if \( x(0) \in \mathcal{R} \) . . .
Example

- unit masses at $y_1, y_2$, connected by unit springs, dampers
- input is tension between masses
- state is $x = [y^T \ y^T]^T$

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
x(t) \\
u(t)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
1 \\
-1
\end{bmatrix}
\]

- can we maneuver state anywhere, starting from $x(0) = 0$?
- if not, where can we maneuver state?

Controllability and state transfer
controllability matrix is

\[ C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 & 2 \\ 0 & -1 & 2 & -2 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -2 & 0 \end{bmatrix} \]

hence reachable set is

\[ \mathcal{R} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \]

we can reach states with \( y_1 = -y_2, \dot{y}_1 = -\dot{y}_2, \) i.e., precisely the differential motions

it’s obvious — internal force does not affect center of mass position or total momentum!
Least-norm input for reachability

(also called *minimum energy input*)

assume that $\dot{x} = Ax + Bu$ is reachable

we seek $u$ that steers $x(0) = 0$ to $x(t) = x_{\text{des}}$ and minimizes

$$\int_0^t \|u(\tau)\|^2 \, d\tau$$

let’s discretize system with interval $h = t/N$

(we’ll let $N \to \infty$ later)

thus $u$ is piecewise constant:

$$u(\tau) = u_d(k) \text{ for } kh \leq \tau < (k + 1)h, \quad k = 0, \ldots, N - 1$$
so
\[ x(t) = \begin{bmatrix} B_d & A_d B_d & \cdots & A_d^{N-1} B_d \end{bmatrix} \begin{bmatrix} u_d(N - 1) \\ \vdots \\ u_d(0) \end{bmatrix} \]

where
\[ A_d = e^{hA}, \quad B_d = \int_0^h e^{\tau A} d\tau B \]

least-norm \( u_d \) that yields \( x(t) = x_{\text{des}} \) is
\[ u_{\text{dln}}(k) = B_d^T (A_d^T)^{(N-1-k)} \left( \sum_{i=0}^{N-1} A_d^i B_d B_d^T (A_d^T)^i \right)^{-1} \]
\[ \times x_{\text{des}} \]

let's express in terms of \( A \):
\[ B_d^T (A_d^T)^{(N-1-k)} = B_d^T e^{(t-\tau)A^T} \]
where \( \tau = t(k + 1)/N \)

for \( N \) large, \( B_d \approx (t/N)B \), so this is approximately

\[
(t/N)B^T e^{(t-\tau)A^T}
\]

similarly

\[
\sum_{i=0}^{N-1} A_d^i B_d B_d^T (A_d^T)^i = \sum_{i=0}^{N-1} e^{(ti/N)A} B_d B_d^T e^{(ti/N)A^T} \\
\approx (t/N) \int_0^t e^{\bar{t}A} B B^T e^{\bar{t}A^T} \, d\bar{t}
\]

for large \( N \)
hence least-norm discretized input is approximately

\[ u_{ln}(\tau) = B^T e^{(t-\tau)A^T} \left( \int_0^t e^{\bar{t}A} BB^T e^{\bar{t}A^T} d\bar{t} \right)^{-1} x_{\text{des}}, \quad 0 \leq \tau \leq t \]

for large \( N \)

hence, this is the least-norm continuous input

- can make \( t \) small, but get larger \( u \)
- cf. DT solution: sum becomes integral
min energy is
\[ \int_0^t \| u_{ln}(\tau) \|^2 d\tau = x_{des}^T Q(t)^{-1} x_{des} \]
where
\[ Q(t) = \int_0^t e^{\tau A} B B^T e^{\tau A^T} d\tau \]

can show
\[ (A, B) \text{ controllable} \iff Q(t) > 0 \text{ for all } t > 0 \]
\[ \iff Q(s) > 0 \text{ for some } s > 0 \]

in fact, \( \text{range}(Q(t)) = \mathcal{R} \) for any \( t > 0 \)
Minimum energy over infinite horizon

the matrix

\[ P = \lim_{t \to \infty} \left( \int_0^t e^{\tau A} B B^T e^{\tau A^T} d\tau \right)^{-1} \]

always exists, and gives minimum energy required to reach a point \( x_{\text{des}} \) (with no limit on \( t \)):

\[
\min \left\{ \int_0^t \|u(\tau)\|^2 \, d\tau \mid x(0) = 0, \ x(t) = x_{\text{des}} \right\} = x_{\text{des}}^T P x_{\text{des}}
\]

- if \( A \) is stable, \( P > 0 \) (i.e., can’t get anywhere for free)
- if \( A \) is not stable, then \( P \) can have nonzero nullspace
- \( Pz = 0, \ z \neq 0 \) means can get to \( z \) using \( u \)’s with energy as small as you like (\( u \) just gives a little kick to the state; the instability carries it out to \( z \) efficiently)
General state transfer

consider state transfer from $x(t_i)$ to $x(t_f) = x_{\text{des}}$, $t_f > t_i$

since

$$x(t_f) = e^{(t_f-t_i)A} x(t_i) + \int_{t_i}^{t_f} e^{(t_f-\tau)A} B u(\tau) \, d\tau$$

$u$ steers $x(t_i)$ to $x(t_f) = x_{\text{des}} \iff$

$u$ (shifted by $t_i$) steers $x(0) = 0$ to $x(t_f - t_i) = x_{\text{des}} - e^{(t_f-t_i)A} x(t_i)$

- general state transfer reduces to reachability problem
- if system is controllable, any state transfer can be effected
  - in ‘zero’ time with impulsive inputs
  - in any positive time with non-impulsive inputs
Example

- unit masses, springs, dampers
- $u_1$ is force between 1st & 2nd masses
- $u_2$ is force between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$
system is:

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -2 & 1 & 0 \\
1 & -2 & 1 & 1 & -2 & 1 \\
0 & 1 & -2 & 0 & 1 & -2
\end{bmatrix} x + \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
-1 & 1 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]
steer state from $x(0) = e_1$ to $x(t_f) = 0$

i.e., control initial state $e_1$ to zero at $t = t_f$

$\mathcal{E}_{\text{min}} = \int_{0}^{t_f} \|u_{\text{ln}}(\tau)\|^2 d\tau$ vs. $t_f$:
for $t_f = 3$, $u = u_{ln}$ is:
and for $t_f = 4$:
output $y_1$ for $u = 0$: 

![Graph showing output $y_1$ for $u = 0$.]
output $y_1$ for $u = u_{1n}$ with $t_f = 3$: 
output $y_1$ for $u = u_{1n}$ with $t_f = 4$: 