TheFourierTransformAndItsApplications-Lecture08

Instructor (Brad Osgood):We're on? Okay. So today – first of all, any questions? Anything on anybody's mind? If you brought your problem sets with you, you can turn them in at the end of the period today; otherwise, put them in the filing cabinet outside my office across from my office by close of business today. Okay. Nothing? No burning issues? No burning questions? Is it the threat of rain that kept people away? It's beautiful out there now, right? Blue skies. Sunshine.

All right. Today, I want to continue down the path that we started last time, of talking about general properties of the Fourier transform. Remember, as I've said now a couple of times, there are two tracks that we're following in developing our understanding techniques of using the Fourier transform. One is to develop specific transforms, transforms of specific functions that you need to have at your fingertips, so to speak, the kind of things that come up often enough that you want to know what that answers are, what the formulas are. And the second path is to understand how to take the Fourier transform and different combinations of functions, different combinations of signals that, again, come up often.

So that's what we're gonna do today, we're going down more the second path, including an extremely important operation. So we're gonna have three big items today, each of which are important in themselves and come up all the time. One is delays, what to do with a Fourier transform when the signal is delayed. One, a formula for what happens to the Fourier transform under a stretch, and finally, a very general operation, which we have now seen a couple times in different forms, but today we're gonna see them today in the context of the Fourier transform in its full glory, so to speak, and that is convolution.

So I want to take each one of these in turn, and I want to derive what the formulas are for you. The book has – the notes have examples of how they're used in practice, and you'll also have plenty of chance to practice some of these in problems. What I want to do is show you what the general formula looks like, and how it comes about.

So let's first look at the question of delay. And there, the question is: If a signal is delayed by – I'm not sure which term you prefer, I'm not even sure what term I prefer, delayed or shifted – say shifted by an amount b, what happens to the Fourier transform? In other words, if the signal is F of t, and that corresponds to a Fourier transform F of s, so I'm gonna use the capital notation here. Then if the signal is delayed by an amount b, and for the purposes of this discussion, that means I consider F of t minus b, not F of t plus b. And, again, it's always a question of what you mean by delay, whether you take a plus or minus, but I think this is probably a fairly standard way of looking at it. Then the question is what happens over here? What happens to the Fourier transform?

Now, this is the sort of question that you have to be able to answer for yourself routinely. These are very simple cases, but there are other cases where similar sorts of things come up, not too many, fortunately. And, again, we have no recourse here, other than to deal with the definition of the Fourier transform. So the Fourier transform of F of s is the integral from -8 to 8, e to the minus 2pi, st, F of t, dt. And then the Fourier transform of the signal F of t minus v is the integral from -8 to 8, e to the minus 2pi, st, F of t minus dt. Now, when you get enough experience, and by now, you probably have that experience, when you look at something like this, the thing that cries out to be done here is a change of variable. You want this to look as much as possible like the ordinary formula for the Fourier transform, that means F of a single variable here rather than F composed with anything.

So what you do is, the simplest thing is to make a change of variable. So u = t - b, b is a constant. So du = vt, and t = u + b then. And if t goes from -8 to 8, since I'm just subtracting off a fixed constant, so does u. So in terms of the new variable, the integral becomes the integral for -8 to 8, e to the minus 2pis, t becomes u + b, and the function becomes F of u, du, in terms of the new variable.

And, now, the thing that's crying out here, is to manipulate the complex exponential. So write that as the interval -8 to 8, e to the minus 2pis, u times e to the minus 2pis times b, F of u, du, and realize that because I'm integrating with respect to u, this second complex exponential here, either the minus 2pi, sb, does depend on u, so it comes out of the integral. It's a constant, as far as integration is concerned.

So I can write that as, e to the minus 2pi, sv, integral from -8 to 8, e to the minus 2pi, su, F of u, du, and what remains is the complex exponential, this complex exponential here, times the Fourier transform of F, the Fourier transform from the original function F. That is, this is e to the minus 2pi, sb, F of s, what I called F [inaudible].

So where do we start? Where do we finish? We started by shifting F, and we discovered what happened to the Fourier transform. In other words, if the original signal F of t has a Fourier transform F of s, then the Fourier transform of F of t - b corresponds to e to the minus 2pi, sv, F of s. With a similar derivation, this is sometimes called the shift theorem, delay theorem, people call it various things; it's very simple and it comes up all the time. And for me, actually, I sometimes have trouble remembering whether it's a plus or minus up here, so that's my own personal burden. So it's actually maybe worthwhile to write down in general F of t, if I change the minus to a plus here, I'd have a plus here. So I even usually write down the formula that looks like this. The Fourier transform of F of t plus or minus b corresponds to e to the plus or minus 2pi, sb, F of s. It's the same formula, I haven't done anything different, but sometimes I like to remember them sort of distinctly. But that's my own burden. This is an example, actually, where the notation can be a little bit of a problem. That is, in this case, sort of the upper case notation is probably most helpful, indicating how the signal is paired with this Fourier transform, and if you change the signal, how does that change the Fourier transform. Because writing down with the operator notation, writing this down, is kind of awkward, because F is already evaluated at t - b, you take the Fourier transform that's supposed to be evaluated another variable. What, do I write like this? Do I write this evaluated at s? That's okay, but that's a notation only a mother would love; it's just too complicated. So we can all understand, sort of by ladies and gentlemen's agreement, what you mean when you write down the Fourier transform of F of t - b. But all I'm saying is that you have to be very careful with your variables and how you write them; what you write, what you don't write. And if you don't believe me, you'll have plenty of occasions to screw it up, trust me, because everybody does. It's just the nature of the subject, that the notation can sometimes be – like I said, there's certain ambiguity to the notation that can sometimes can cause problems. So this is sort of a bad notation in that case, although, I don't usually like to use it as probably a better notation. So the formula itself is very simple. What is the interpretation of the formula? And the interpretation is also something to keep in mind, and that is a shift in time corresponds to a phase shift in frequency. So here, I think it's actually better to use the word shift instead of delay because I can use it in both sides of the sentence. Shift in time corresponds to a shift in frequency, but a shift in frequency you think is a phase shift, corresponds to a phase shift in frequency. If I think of the signal as living in the time domain and the Fourier transform as living in the frequency domain. Phase shift in frequency.

So why do I say this? Well, because what I have in mind, with a statement like that is, that the Fourier transform is a complex number, so it has a magnitude and it has a phase. So I write F of s, the Fourier transform of the original signal as its magnitude, the magnitude of F of s, that's a real number, times e to the 2pi, say, theta of s. So theta is the phase. And then if I multiply – this is not a hard thing but it's a thing you have to keep in your head, not only the formula but any interpretation of the formula. So e to the 2pi or e to the minus 2pi, sv, times the Fourier transform of s is – the magnitude stays the same. And it's this complex exponential times that complex exponential. So that's e to the 2pi theta of s minus 2pi, s, theta of s - b. I have an s in both cases. No, I don't, 2pi theta of s - sb. In general, the phase can depend on s. That's why I write a theta of s, the phase can vary with s as well; it usually will. So there's the phase shift. The magnitude is the same. The magnitude of the spectrum does not change. The magnitude of the Fourier transform does not change. What changes is the angle. So I will let you push that interpretation however far you want, that is if you want to think about other sorts of ways, but that's the basic interpretation of the shift theorem. A shift in time corresponds to a phase shift in the frequency and the magnitude stays the same. So, again, I will let you convince yourself if that makes sense on physical intuitive grounds, if you think about a signal and how it's made up. Next, I like these formulas, and I think it's nice the way these formulas work out. This is not as exciting as some of the other things we've done, I admit. But, nevertheless, it's part of the whole package that you have to be familiar with; you have to be comfortable with. And it's not only the formulas, it's how the formulas are derived. One of the reasons why I'm going through this is just to show you, and I hope you can also show yourself, is just the kind of techniques that are involved in deriving these formulas. Because if you can see them in these relatively simple circumstances, often it's the case that you can apply them in more or less the same way in slightly less obvious or less simple circumstances. It's a pedagogical kind of thing. Now, the second general theorem, or formula, is what happens when you scale the time. And here, actually, there is a very important interpretation that I'll talk about after we get the formula, but this one I think reveals a deeper property of the Fourier transform.

Scaling is scaled – when I say scaling, I'm talking about scaling the independent variable of scaling of the time. If F of t, again, corresponds to, let me write it like this, the Fourier transform F of s, and then if I scale the time by a constant a, so I'm thinking of a as just a real number here, but not necessarily positive, then what happens to the Fourier transform? So, again, to answer this question, we have no recourse in terms of the definition. So the Fourier transform of a scale version of F is the integral from -8 to 8 of e to the minus 2pi, st, F of at, vt. And, again, the thing that is crying out to be done here, is a change of variable. But it's helpful, I think, to make a distinction here, whether a is positive or negative, because of what happens when you make that change of variable to the limits. You know, we have [inaudible] definite interval, the limits of integration also have to change, and it's helpful here to make a distinction. They change differently whether a is positive or negative. So let's take the case when a is positive first. So if a is positive, I'm gonna make, in either case positive or negative, I'm gonna make the change of variable to du = a times dt, that's clear that I want to do that. Then, of course, du = a times dt. Now, if a is positive, and t goes from -8 to 8, then so does u; u also goes from -8 to 8 because a is just a fixed constant here. So the integral becomes the integral from -8 to 8 of e to the minus 2pi, s, u/a times F of u; dt becomes 1/a times du. The a is a constant; I can factor that out of the integral. The integral is from -8 to 8 and this is minus 2pi. I'm gonna put the 1/a with the s instead of the u. And if I do that, why it's plain enough what to see, what remains is the Fourier transform of the original signal F, evaluated not at s but at s/a. So that is 1/a times the Fourier transform of the original function, not evaluated s but evaluated s/a. Fine. Done.

Now, what about if a is a negative, let's take that case. So, again, I'm gonna make the same change of variable, du = a times dt, du = a times dt. Formally, that's all the same. Now, what happens to the integral? So here you have to be a little more careful. If a is negative, and t goes from -8 to 8, then a times t goes from +8 to -8. Some people are skittish when I write integrals. They say, "You can't do that, you can't write an integral from a bigger number to a smaller number." And I say, "I just did. Nothing happened. God did not strike me dead." So you just have to be careful. So it's the integral from +8 to -8, and then the rest of the integral looks the same formally, that is minus 2pi, s, u/a, F of u, and then dt becomes 1/a times du. So the 1/a comes out of the integral. I'll do this in two steps. So it's 1/a, the integral from +8 to -8 of e to the minus 2pi, and, again, I'll bring the 1/a to the s instead of u, F of u, du. And now, if I want to recognize this as a Fourier transform, I swap it with limits of integration, instead of integrating from +8 to -8 I integrate it from -8 to +8, but then of course I have to change the sign. So this is minus 1/a times the integral from -8 to 8 of either the minus 2pi, s/a, u, F of u to u, which is minus 1/a times the Fourier transform evaluated at s/a, which is minus 1/a times the Fourier transform of the original signal evaluated at s/a. Now, you can combine these two. So that's how the two formulas look. And they're different, sort of, when a is positive and when a is negative. But, really, you can combine both cases if you realize that if a is negative, minus a is positive. In other words, this is the same thing as 1 over the absolute value of a times the Fourier transform of s of a, the Fourier transform from the original signal evaluated at s/a. And, of course, this also includes the previous formula because if a is positive taking the absolute value doesn't do anything to it, it just returns to the original value. In other words, the stretch theorem or the similarity theorem, whatever term you want to use, can be written in one piece. I had to the derivation, really, in two parts, but there's only one formula. And the formula is, that if F of t corresponds to

F of s, then F of at corresponds to 1 over the absolute value of a times the Fourier transform of the original signal at s/a. So notice the absolute value is on the outside here but not on the inside.

Now, suppose if you should say something like, of course a is not zero, but – you know, I take triviality insurance out on all my statements. If my statements are trivially true or trivially false, you can ignore them, because I know I will. Okay. That's sometimes called the stretch theorem or the similarity theorem, because the scaling is either called stretch or a similarity. Similarity is probably a better term because stretching and shrinking can be the same sort of operation, depending on whether or not a is bigger than 1 or a is less than 1. And let's talk about that point. What is the interpretation of this? Because this actually has, I think, a more fundamental interpretation than the stretch theorem. What is the interpretation? Well, let's take the case, first of all, when say, a is bigger than 1. So I'm gonna just consider a positive here. So I'll take a to be positive, and then first take a bigger than 1. If a is equal to 1, I'm not doing anything, of course. If a is bigger than 1, then how does, say, the function F of at compare with the function F of t or the graph of the function F of at compare with the graph of the function F of t? So how does F of at compare with F of t? Well, if a is bigger than 1 then – you saw this back in eighth grade, when you were first learning these things, but sometimes I have to give myself an argument for how these things look like. So if the signal is sort of like this, if this is F of t, then F of at is the same signal except it's squashed; it is compressed. Same shape, more or less, but it's sort of a squashed version. F of at is a compressed form of version of F of t. You convince yourself of that, that if the variable is bigger, like if I scale by something bigger than 1, then I'm squeezing the function. What about Fourier transform, what happens on the Fourier transform side? On the Fourier transform side, originally I had the Fourier [inaudible], and then I go to the Fourier – if I scale it, I go to 1/a, F of s/a. Now, I'm gonna draw the graph, but the graph is a little bit deceptive here because you can't draw the graph of the Fourier transform because the Fourier transform is complex value. People make this slip all the time, where they think they're gonna draw a picture of the Fourier transform, but you can't really draw a picture of the Fourier transform because you'd be plotting complex numbers. So what people actually plot, when they plot pictures of the Fourier transform, is its magnitude. So when I draw a picture here make that a realization. So if this is the original Fourier transform, if the signal is real, the magnitude of the Fourier transform is gonna be symmetric, so it's gonna look something like this. It looks a lot that, actually, but that's just because I have lack of imagination when I draw my functions. So if this is F of s, or rather say the magnitude of F of s, then if I scale here what happens? If a is bigger than 1, 1/a is less than 1. So I'm scaling s by something less than 1. If I scale it by something less than 1, I am stretching the graph out instead of compressing it, and I'm also multiplying by something less than 1 out front. So there's actually a stretching in the horizontal direction, and a shrinking in the vertical direction. And, again, this is not so easy for me to draw, but it's like a stretched out version of the signal, and shrunk.

So it's stretched out horizontally, so this 1/a, I don't have to put the absolute value there because I'm assuming it's positive, 1/a F of s/a, magnitude. So it's stretched out horizontally, and it's also squeezed down vertically. Is that right? How do I say it, decreased vertically? Squashed, those are the technical terms, squashed vertically. So that's what happens if a is bigger than 1. If a is less then 1, the situation is reversed. If a is less than 1, then if this is F of t then F of at is stretched out. And what happens to the Fourier transform, that is squeezed together. Because, again, the Fourier transform is then scaled by something bigger than 1. So, again, the original Fourier transform goes to 1 over the absolute value of a, a is positive, 1/a, F of s/a. But, again, here if a is less than 1, 1/a is bigger than 1, so the Fourier transform is squeezed. So if this is the original Fourier transform, so to speak, absolute value of F of s, then 1/a is gonna stretch things in the vertical direction, because 1/a is bigger than 1 and it's gonna squeeze things down. So maybe something like that. So it's squeezed and stretched. Now, think of the extreme cases here. Think if a is getting bigger, bigger, greater, greater than 1 or if a is getting less and less than 1, what does it mean? Let's take the case when a is bigger than 1, much bigger than 1, it's getting larger and larger. If a is getting larger and larger, then the signal is getting more and more compressed, it is getting more and more localized in time. On the other hand, what is happening to the Fourier transform? If a is getting larger and larger, the Fourier transform is getting more and more stretched out, and for that matter, also squeezed down in the vertical direction.

This is an example, an extremely important example, of this sort of reciprocal relationship. And I use that term pretty broadly here. I mean, you see reciprocals coming in, but sort of in words and in feeling you can't have a signal, which is both localized in time and localized in frequency. To localize the signal in time would be to squeeze it down. Like, look at the scaling of this F of at for a very large a. But that has a consequence and frequency of stretching things out. So if you want to concentrate the signal in time, the Fourier transform is gonna stretch out. And the reverse situation also holds. If you take a signal and stretch it out in time, by scaling by a number less than 1, you know, scale by 1/10, by 1/100, by 1/1,000, whatever, then you are stretching it out in time, but you're concentrating it in frequency. The Fourier transform is getting squeezed. That's a very important and a fundamental property of Fourier transform, of the picture and the signal, of the signal in the time domain and in the frequency domain, and it comes up all the time. It's a very important piece of intuition for you to have. And to me, that's somehow the most important consequence of the stretch theorem. This theorem is what it says about localizing the signal in time versus localizing the signal in frequency. A signal cannot be both localized in time and in frequency. This is one instance of this. There are other instances, actually, there are other ways of writing down that sort of relationship, of translating this sentence into more precise mathematical terminology.

One of the most famous ways of translating this is the Heidenberg Uncertainty Principle, that you can't localize a particle both in space and in time. You can't know both it's velocity and its position to arbitrary precision. The Heidenberg Uncertainty Principle can be proved, and there's a derivation in the notes, actually, by taking you through the Fourier transform, by using the Fourier transform. And it essentially comes down to this fact, that a signal cannot be both localized in time and in frequency. That if you stretch out of one domain you're gonna squeeze in the other domain, and vice versa. This is a very important point of intuition for you. Now, one interesting thing, if you think about what we showed about the Gaussian, we showed, as a remarkable fact, that the Fourier transform of the Gaussian is itself. So recall the situation with the Gaussian, F of t = e to the minus 2p t squared. When it's normalized this way, the Fourier transform is the same thing, F of s = e to the minus p s squared, same thing. So from interpreting the Gaussian, or putting this interpretation onto the Gaussian, you might say the Gaussian is perfectly balanced somehow in both time and frequency. It is what it is; it is perfectly balanced. It's just concentrated enough in time, and just concentrated enough in frequency or just spread out enough in time, and just spread out enough in frequency. So it's the same thing, taking the Fourier transform doesn't change it. So it's perfectly balanced. I don't know how better to say this, perfectly balanced in time and frequency.

So these things are in general use now and everyday use. You're gonna use these things all the time. People that work in signal processing can just whip these formulas out without any problem, and they are quiet familiar with these interpretations. Well, the formula that I won't write down for you, but is, again, in the notes, and there's some examples of it, and I think there's some problems of it, are what happens when you combine shifts and stretches? The formulas get a little uglier, there's nothing new involved in that, but it's just a question of combining the formulas in the right way. And part of it, actually, is a question of the variables, and how you understand parenthesis and everything else. So I'm not gonna write down that formula, because unless I practice it a couple times, I'll get it wrong. But that's the one sort of formula in this general circle of ideas that also comes in is, the formula for the Fourier transform when you both stretch and delay, so F of at + b, something like that. So I'll let you struggle with that quietly.

So one more, go out with a bang today, and that is to introduce the idea of convolution. Again, something that you've probably seen in other classes, something that we've already seen in the context of the Fourier series. But now we want to see convolution in the context of Fourier transforms, and the way that it arises in the context of Fourier transforms, which is different, most naturally different, than the way we saw it first coming up in the context of the Fourier series. There we saw it in the context of solving differential equations, and that same interpretation, or that same use of convolution, again, comes up. But that's not the way it's motivated, and that's not the most fundamental role it plays, at least for the questions of signal processing. So convolution is probably, I think it's probably safe to say, it's probably the most important operation in signal processing. I think that may be a little bit of an exaggeration, but not too much. So I'll say probably. Most frequently used, most flexible important operation in signal processing. Now, what is signal processing, since I've used that phrase, or what is the fundamental question of signal processing? Signal processing, very broadly, can be said to – you might define it as saying, how can you use one function to modify another? How can you use one signal to modify another? That's the basic question of all signal processing. Don't lose sight of this, is how to use one function to modify another or one signal?

You can ask this question or you can take this approach in either the time domain or the frequency domain. And, again, it's equivalent because we can go from one to the other by the Fourier transform, or the inverse Fourier transform. I'd say it's probably more common to address this question in the frequency domain. That is to say, when you talk about modifying a signal, most often, not always, but most often you're talking about modifying the spectrum of the signal. So most often you look to modifying the spectrum of a signal. And then at the end of the day, what it is you take the Fourier transform, you screw around with the spectrum, that's the technical term for it, and then you take the inverse Fourier transform and you modify the original signal somehow. But the operations you perform are often conceived of and carried out in the frequency domain, and then you go back and forth between the two domains, again, by the Fourier transform or the inverse Fourier transform. Now, there's a very simple example of this. For example, linearity actually gives you, in some sense, that's the simplest example of signal processing, or simplest operation of signal processing, e.g., linearity or super position. That is the Fourier transform of F + G is the Fourier transform of F plus the Fourier transform of G. So think about F as the signal that you want to modify; think of the signal G as the one that's doing the modifying. If you don't like the spectrum of F, fine, add something to it that takes away a little bit here, it takes away little bit there, adds a little here, adds a little there, and then go back. So you modify F, F, the spectrum of F, by adding the spectrum of G. You don't know what G is, but then you can take its inverse Fourier transform and find out the function that does the modifying for you. Usually, think of linearity as sort of the simplest property of Fourier transform that follows from the linear to the integral, the integral to the sums, the sums to the integrals, and you don't think it's really worth commenting on. But in some sense, it's the simplest and most basic operation of signal processing; you don't like the spectrum, add something to it. Viewed in that light – and of course I can also scale by a constant, that's even less interesting in some sense. That's the other part of linearity is I can multiply by a constant here. But, anyway, viewed in this light, a very natural question is: What if I want to multiply? So what about multiplying? That is to say: On the frequency side, I have my original function F and I want to multiply it by this Fourier transform and other function G. So I'm scaling each individual frequency, so to speak, each individual point in the spectrum by some function G that I want to construct, that does what I want it to do. And the question is: How does that come about? That is, is there some combination of F and G in the time domain, so that in the frequency domain, so within the spectrum, the spectrums multiply, the spectrum of F is multiplied by the spectrum of G. The values of the Fourier transform of F are multiplied by the values of Fourier transform G. Can I do that? Is this F of some F combined in some unnatural way with G? That's the mystery. Can I find a way of combining F and G so that the sequence of the Fourier transform is multiplied? Not so unnatural. Not so unnatural. So to answer this question, we're going to adopt a now familiar strategy. You've heard me say many times, this is how mathematics gets applied. Suppose the problem, solve, and sees what has to happen. So suppose the problem is solved – now, I want to be careful, I want to look at my notes because I want to use the same variables that I use in my notes so I don't confuse the issue, if you go back and forth between what I'm saying in class and what I say in the notes. So like I said, suppose the problem's solved? So I look at the Fourier transform of G at s times the Fourier transform of F at s. And I just play with this. What happens if you multiply the two spectrums? Well, the Fourier transform of G is the integral from -8 to 8. I have no recourse here but to go to the definition. Then -8 to 8 either to the minus 2pi, s of t, st, G of t, dt. And the Fourier transform of F at s is the integral from -8 to 8. And here I want to write this – I already used t, integrative perspective t and so I want to integrate with respect to another variable, and you'll see why in just a second, e to the minus 2pisx, F of x, dx. All right, I think I can take it from here. That's all I wanted to check.

Now, watch this. This is the product of two single integrals, they are decoupled, so to speak. So that is I can write this as one big double integral, e to the minus 2pist, e to the minus 2pisx, G of t, F of x, dt, dx. Okay. There should be no problem, you should be able to see that. And if you're having trouble seeing it, instead of going from here to here, imagine yourself going from here to here. The variables are decoupled. When I say the variables are decoupled here, I mean they're decoupled. I don't know how else to say it. There's no relationship between the variables; everything happens independently. Now, combine the complex exponentials. That is, write this, integral from -8 to 8, integral from -8 to 8. By the way, there are issues here. I mean, there are things about turning – I'm gonna swap limits of integration, I'm gonna swap dtdx, dxdt, all the rest of the rest of that stuff. As I have said before, the rigger police are still blessedly off duty. All right. So this is, e to the minus 2pi, I have a common factor of s in both exponentials, and I get a t + x. G of t, F of x, dtdx. I haven't done anything there except algebraically combine the two exponentials. Now, I want to do something a little trickier, I'm gonna write this double integral as integrated single integrals. So now the variables are coupled, really. So I'm gonna write this as the integral from -8 to 8, the inside integral, integral from -8 to 8, e to the minus 2pis, t + x, G of t, dt, and then the result is integrated against F of x, dx. Now, let's check to make sure that everything's okay here, that is actually get a number at the end of the day.

On the inside integral I have a function – this kernel here involves both, well, st and x. I'm integrating with respect to t. So what remains is a function of x, forget about the s that comes on sort of in the end, but think about this as a function of t and x because those are my variables of integration. So I've integrated this with the respect to t, this inside integral. What remains is something that depends on x. I multiply that by F of x. I integrate with respect to x. I'm done, everything's okay. So the only thing that remains here is the function of s, and that's the product of the Fourier transform. So that's what you expect. Now, once again, what this calls for, if you can only hear it calling, I'm training your ears and your eyes, is a change of variable. So I'm gonna let u = t + x, so du = dt. See, the inside integral is with respect to t, so that's the variable here, u = dt, and then t = u - x. So what happens to the inside integral? And, again, if t goes from -8 to 8, for any fixed x, so does u go from -8 to 8. So I get the integral from -8 to 8 of the integral from -8 to 8 of e to the minus 2pis, t + x is u, G of t, which is u - x, du then integrate with respect x, F of x, dx. Cool. It may seem pointless, but it's not. Now, I'm gonna swap the order of integration. Instead of integrating du, dx, I'm gonna integrate dx, du. And I'm gonna write this as the integral from -8 to 8. The inside integral becomes the integral from -8 to 8 of G of u - x, F of x, I'm skipping a step here a little bit, but not much, times e to the minus 2pisu, du. Is everything okay there? Let's check. I skipped a step here, I sort of put everything together. I could have taken one more step and put everything together, and then pulled it apart, so I have, again, the two integrated single integrals. This integral with respect to x, and then what remains – again, I integrated this with respect to x, what remains is the function of u. I integrate that against this kernel, this complex exponential, e to the minus 2pisu, with respect to u; what remains is the function of s. Want me to say that again? No. All right. So I skipped a step there. I took that integrated integral, I could have written everything together, grouped everything together, and then split it up again into to integrated integrals. All right. Now, look, say, we have eyes if only we but see. This integral inside here is a combination of G and F, and is getting integrated against a complex exponential. That is, if I now am so bright, I'm so farsighted, if I am so brilliant, I will now realize that I have solved my problem. I'm gonna define h of u to be the integral from -8 to 8 of G of u - x, F of x, dx. Right? That's what happened to the inside integral, u - x, yeah, F of x, dx. And then what I have is the integral from -8 to 8 of either the minus 2pisu, h of u, du, which is the Fourier transform of h evaluated at s.

Now, once again, where do we start? Where do we finish? I discovered that the Fourier transform of G at s times the Fourier transform of F at s, is the Fourier transform of h at s. The spectrums multiply if I define h by this formula. If we define h of u = the integral from -8 to 8 of du - x, F of x, dx. I solved my problem. Once again, what was the problem? I said is there a way of combining F and G in the time domain, so that in the frequency domain the spectrums multiply, the Fourier transforms multiply? And the answer is, yes, there is, by this complicated integral. It's not so obvious. You wouldn't get out of bed in the morning and write this down, and expect that this was going to solve that problem. How do we get it? You said, suppose the problem is solved, see what has to happen, and then we'd have to recognize when it's time to declare victory. And it's time to declare victory.

Now, being an obnoxious mathematician, you cover your tracks and say, "Well, I'm simply going to define the convolution of two functions by this formula." Let me use a different variable here to write the definition down. That's how mathematics works, you turn the solution of a problem into a definition. It irritates the hell out of some people, but that's how it works. So define the convolution of F and G by this formula, say and the special notations use G * F. Well, let me call the variable x here. So it's the integral from -8 to 8, G of x - y, F of y, dy. Define the convolution of that? And then if you're a mathematician, you'd say, "Well, sure, I've defined that. What can be more natural." And then you've proved the theorem. Then you proved separately, you proved this remarkable fact, that the Fourier transform of the convolution is the product of the Fourier transforms. But what's missing in that derivation is, the game is rigged. I mean this definition was come upon precisely so it would satisfy that property, so that the Fourier transform multiplied the two signals, so the Fourier transforms multiply.

Then we have the Fourier transform of the convolution, G * F is the Fourier transform of G times the Fourier transform F. Remarkable. But as I said, the game's rigged. I mean, I set out to solve that problem, and I solved that problem, and then I sort of covered my tracks by saying, "Well, define the convolution by this, and then we have the remarkable convolution theorem, that the Fourier transform of the convolution is the product of the Fourier transform." But of course you do because you set it up that way.

Now, I don't want it to seem like we've done something insignificant here, we've done something extremely significant. Because the idea of being able to modify a signal by multiplying the Fourier transforms is a very powerful thing to be able to do. And what this says is, there is a way of combining the signals in the time domain by a not so very obvious thing. It's a nice enough looking formula, it's sort of an elegant looking formula, but it's not obvious that this would work. And if you combine the signal this way in the time domain, in the frequency domain the Fourier transforms multiply. How cool is that.

All right. And I think on that note of coolness, we'll wrap it up for today, and then next time I'll talk about some of the properties of convolution and why it is so important to your everyday lives. See you then.

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Duration: 51 minutes