TheFourierTransformAndItsApplications-Lecture13

Instructor (Brad Osgood):On the air. Oh, what these people miss before the camera starts rolling. Okay. I sent out a note yesterday, or over the weekend about the mid-term coming up. So, the midterm is coming up a week from Wednesday, and without going into too much detail right now, we're going to have three sessions – it's a 90-minute exam outside of class, so we'll have class that day.

I am scheduling in three sessions: 2:00 to 3:30, 4 to 5:30, and 6:00 to 7:30, so there's a sheet circulating around, I'm asking you to sign up for one those times. This is not a contract you're signing, all right? This is just so I have an idea the relative sizes of the different sessions are, so I can find an appropriate room. And I'm hoping by having three of those times that that should pretty much take care of everybody. If there are people who can't make it any one of those times, then please let me know, we'll make some special arrangements for it. I'll have more details about the mid-term as we go on later in the week. But the general idea of it is that – I don't want it to be too computationally intensive. That is to say, I'm trying to make the exam, sort of, a little bit more on the conceptual side, so you don't get bogged down in integration by parts and things of – you can't avoid all computation of course, but the way we're going to try to write the exam is to try to keep it on the conceptual level rather than detailed calculations. And let's see, it'll be open book/open notes. Now, sometimes people ask me if they can bring any book they want and I say, 'Yeah, you can bring any book you want.' But this leads to ridiculous events where I've seen, you know, I've seen students walk into the exam with, like, a stack of 10 "Signals and Systems" books, now, which is just ridiculous. But, if you want to, hey, that's okay.

I'll provide the formula sheet; that's already posted on the web, and the blue books and all the rest of that jazz. And let's see, I've posted already also last year's mid-term exam plus the solutions, so that's there on web. And also, the most recent – the next homework assignment is also up. Okay. Any questions about anything? Any questions or comments? Nope. Okay. Big day today. Big day, big day. It's finally time to reap some of the benefits of the discussion, rather general, and I think it's safe to fair say, abstract discussion we've been having. It'll still – it may still seem a little abstract to you today, I understand that, so just ride along, all right? And you'll see today some amazing formulas that come out of this effortlessly in finding the Fourier transform of some well-known functions – things we're really gonna have to use. It's really just, I don't know, a bucket full of miracles and the fun never stops. So, I don't want to talk today about the Fourier of transform of a generalized function, or a distribution, also known as distribution Fourier transform of a distribution. So let me remind you, first of all, what the setup is, what goes into this. To define a distribution you need a class, first of all, a class of test functions. So the setup is, you first need – you first have to define a class of test functions, or test signals that usually have particularly nice properties for the given problem at hand. And it can vary from problem to problem. For us, for the Fourier transform, it's the class of rapidly decreasing functions. So these typically have particularly nice properties.

Sorry for not specifying that terribly carefully. But they come, generally, out of – again, sort of, years of bitter experience with working with problems, working with a particular class of applications and trying to decide what the best functions are for the given class. For Fourier transforms, the class of test functions is the rapidly decreasing function, so I won't write down the definition again, but I'll remind you of the main properties in just a minute when we need it. Rapidly decreasing functions – these are the functions which are infinitely differentiable and any derivative decreases faster than any power of x, so they are just as nice as you could be. Rapidly decreasing functions. All right. That's what you need. You need a class of test functions, and then by definition, a distribution or a generalized function is, to use the most compact terminology, a continuous linear functional on the set of test functions. Distribution, also called a generalized function, this term is probably not used – generalized function is probably not used as much as it once was when the subject was new. Nowadays, people just call it distribution since Schwartz is really unifying work on all this. Is – so let me give you the shorthand notation for this, a continuous linear functional on test functions. All right. So that means it satisfies two properties. So one, so you write the pairing if is a test function, and t is a distribution, we will write – most of the time, but not all of it, not always t, , with that sort of pairing for t operating on .

All right. So if you want, you can think of as some distribution of physical information, temperature, voltage, current, whatever; and t is a way of measuring it. All right. So you measure something, you get a number. All right. So t is a measuring device and is the thing you are measuring. So t operates on in such a way that it satisfies the principle's super position, or that is to say, it's linear, and it's also continuous. So t is linear, meaning t of 1 plus 2 is the same thing as t operating on 1 – I'll write up here – t operating on 1 plus t operating on 2. And the same thing for scaling. So t – apply – t operating on alpha times , is alpha operating on t of . That's the property of linearity. And the property of continuity can be phrased in different ways, but the most direct way of writing it – and I'll have a comment about this in just a second – is that if n converges to , a sequence of test functions converging to a test function of , that implies that the measurements of n also converge to the measurement of the final test function.

So, really, there's nothing wrong with trying to impart, sort of, a physical interpretation, but I don't wanna try to push that too far. But if you think of as some sort of something to be measure and t as the thing that does the measuring. Now, I wanna make a comment about the mathematical – the one – one quick comment about the mathematical for the foundation for this. The hard work in all of this is this statement right here – n converges to v. All right. The hard mathematical work is to define n converging to , what that means. Because these space of functions, although they – each individual function has particularly nice properties – infinitely differentiable, all the derivatives decay, and so on and so on – if you want to talk about a sequence of functions converging, then the more properties you insist a function having, the harder it is to control the sort of convergence. You know, if the function is infinitely differentiable then you want all the derivatives to also to converge. If the function is rapidly decreasing, you want all the derivatives to be rapidly decreasing, and so on. So it's hard to write conditions down that are gonna control this, that's what's difficult to do. And if you open a math – this is a warning – if you open a math book that says something like, ‘theory of distributions or theory of generalized functions – it’s gonna be, like, the first hundred pages or so, are gonna be devoted to analyzing what it means to say that n converges to . And you look at that and you’d say, ‘Why would anybody learn this subject?’ and I wouldn’t blame you for saying that. All right. It’s complicated, there’s no two ways about it. If you’re gonna give a very precise treatment of it, then you really have to do that. We don’t have to do that, all right? I’m gonna take this sort of, again, as intuitive as you can make that. Imagine controlling the functions, all the derivatives and so – you can think in terms of the graphs, the graphs of n converge to the graph of , and everything is controlled as nicely as you want. So suffice it to say, it can be defined precisely, but that’s not for us to do. All right. But I did want to point that out.

The other bit of terminology here for those of you who have seen this, for those of you have studied a certain amount of extra math, and particular the field of – it’s usually called functional analysis – is that one says that the distributions are – the set of distributions is the dual space of the set of test functions. That’s another way of phrasing this. So the distributions – the particular class of distributions that you’re considering are the dual space – is, are – dual space of the space of test functions. Bring that up at a cocktail party some night, watch the people scatter. All right. Okay. Now, we had a couple of examples last time and I want to remind you of that. The first example we had was delta defined as a distribution, and it’s the simplest distribution. This mysterious delta function that’s supposed to capture this property of concentration at a point emerges as simply evaluation. And so examples: delta. All right. So, delta as a distribution is defined simply by delta operating on is of zero. All right. This replaces – now that’s a pairing – once again, in all this what you’re going to hear me say a lot today is, I’m gonna define a distribution. What does that mean? That means, you give me a test function, I have to tell you how to operate on it. All right. So you give me a test function of , to define a distribution, I have to tell you how that distribution operates on . So I say, ‘How does delta operate on ?’ Delta operating on is just of zero, nothing else. Period. All right. This makes precise – this dispenses with all of that bullshit about the classical – the classical definition of delta is zero everywhere, except at one point where its infinity and the interval is one, and so on and so on. All that is gone, all that is swept aside by this definition. Now, you know, it took awhile to get to that definition, okay, but once its there, that’s what you can work with. Okay. Likewise, there’s the shift of delta. I called it delta sub-a, and that’s just defined by – so, again, to define a distribution you give me a test function, I have to tell you how it’s defined. I have to tell you how operate. And I say, delta a operating on is of a. Period. Period.

Okay. Now, I would be remiss if I didn’t indicate – there’s a graphic way of picturing these deltas, all right? So the graphical picture – and we’ll use this too – usually indicate the delta function by a spike at the origin, delta, all right. And since it’s supposed to have infinite height, I can’t exactly draw an infinite spike, so I just put a little arrow there. Everybody writes it like this, and likewise, delta sub-a is usually indicated by a spike at a, okay? Now, something else. All right. Since we’ve been so careful about this, and since I’ve made such a big deal out of the fact that it’s all airtight and we can dispense with intervals and things like that, I am not going to blame you and I am not going to draw –

I'm not going to try to derive out of your system, expressions like this. Interval for minus infinity to infinity of delta of x, of x, x, is equal to of zero. If you want to write that, okay. I'll even write it from time to time. Its okay, nothing bad will happen to you. All right. Among friends, it's perfectly okay. And a corresponding thing with a shifted delta function, it's all right. All I want you to be aware of is that one can – this is just really a mnemonic in some senses – this is in some way a more complicated way of writing that, all right? But in some formulas and in some contexts it's helpful to write that and, okay. All right. Do not be afraid now that you have the hidden knowledge. All right. What's really going on because this is what's really going on? The second example we had of distribution was really an example of the whole class of distributions, and that was by way of assuring us that nothing has been lost. All right. Now, these things are sometimes called generalized functions, so if they're generalized functions they ought to include the original functions, the good old functions. And they do – that is the second example, we say a class of examples – are really distributions induced by – this is probably the best way of saying it – induced by functions. All right. So if of x is a function such that I can integrate it against the test function, such that this interval minus infinity to infinity, say of x, of x, x makes sense, the interval converges. All right.

And that will be the case for many functions because is such a nice function; may be such a nice function that, even if is a bad function, the product is gonna be such that the integral converges. All right. If that makes sense, then that's a number after all, I integrate of x against of x and that's linear, as the sum of the integral, the integral of a sum and so on, and it's also continuous. That's a little harder to show, and so that defines – that defines a pairing of and . Continuous – again, I'm not gonna show that a pairing of and . In that sense, defines or reduces a distribution. You give me a test function; I have to tell you how the distribution operates on it. How does it operate on it? It operates by that pairing. And so for shorthand, I write paired with – what I'm really thinking of here is the distribution induced by , but that's' too many words. So of x times of x would be x. Okay. So a lot of functions which are not as nice as the test functions are included in the set of distributions instead of operated – functions – things that operate on test functions. So for example, it may be the case that one pairs with , so this includes many functions. Many functions induce distributions. So it can be regarded as distributions. So e.g. – it may be the case that integral of one minus infinity to infinity, of x, x, which is the integral for minus infinity to infinity of one times of x, x – that may make sense. It should make sense; you don't want a function which is not integral for goodness sakes. And so in that sense, one reduces the distribution because I can allow one to operate on by means of this integral. Okay.

And likewise, signs and cosigns are not themselves integral, but if I multiply them by function, which is really dropping down, which is decreasing at plus or minus infinity, then the integral of sign or cosign against such a function will make sense. And in that sense, sign and cosign can be considered as distributions, that is they operate them. So let me – well, just for short, the complex exponential often works. So either the two pi i a x is – might be okay – provided the interval for minus infinity to infinity – either the two pi i a x, of x, x makes sense. Which for many functions it will. Okay. It will, even though either the two pi i a x itself is not integrable, if I multiply it by a function which is decreasing, the product will be integrable, say. All right. So that induces a distribution. How does it operate on the function? It operates on the function by integration. Okay. All right. And if you look at this example and if you think of a Fourier transform coming in, you wouldn't' be too far off, because now the big moment has arrived where I'm going to define the Fourier transform distribution. So we talk about distributions and how you'd – the definition of distributions and a couple of examples of distributions, now I wanna do something with them. And this is so cool how this works, I think. Again, just enjoy the ride. All right. And watch how these derivations go. It's almost effortless the way this all works. I mean, it's just so – it's better than sex; not that I would know, I was a math major, but –

So I want to define the Fourier transform of a distribution. So now we're gonna look – the test functions we're gonna look at now are the Schwartz functions. The rapidly decreasing functions. So let's take the test functions to be s – the rapidly decreasing functions, all right? Now, let me remind you of why they are so good Fourier transforms, all right? Why they are so good for Fourier transforms is if is in s, then so is its Fourier transform. All right. And for that matter, I guess I don't think I said this, but for that matter, the inverse Fourier transform is also an s. Okay. An inverse Fourier transform differs from the Fourier transform only by a minus sign, so if one isn't there, it's easy to see the other one is. So that's one property. And the second is that Fourier inversion works. All right. It is in that sense – those sense – let me write the second one down. That's the first property is that the function is rapidly decreasing, so is it's Fourier transform. Second is that the inverse Fourier transform of the Fourier transform of is equal to and, likewise, the Fourier transform of the inverse Fourier transform of is equal to . Those are the properties that make the right class, the best class, of the classical functions to use when you want to talk about the Fourier transform. The classical Fourier transform is defined by the integral. All right.

So these are the functions we're gonna use. The corresponding class of distributions is sometimes called the class of tempered distributions. Distributions are called the tempered distributions. All right. And what I wanna do now is I wanna show you how to define the Fourier transform of a tempered distribution to be another tempered distribution. Okay. If t is a tempered distribution, I want to define its Fourier transform. And this will be another tempered distribution. Okay. Now, again, you will hear me say over and over again today, I want to define a distribution. What does that mean? That means you give me a test function, I have to tell you how it operates. So I have to tell you how to define the Fourier transform, something I'm gonna call the Fourier transform of t operating on , or paired with . So I have to define the Fourier transform of t operating on , where is a test function – rapidly decreasing function. What shall I – how shall I do it? How shall I do it? All right. There is a guide in this. There is a guide in all things, all questions of this type. You wanna know how you should define an operation on a distribution, you ask yourself – you start off by asking yourself, 'What would I do if I were in a really good situation here?' Like, if of t were itself a rapidly decreasing function, say, or some other good function and this pairing was my integration. All right. So what if? This is another example of suppose the problem is solved, let's see what has to happen. All right.

So what if the pairing is by integration? It may not be in general, okay? I mean, delta doesn't rise by integration, but suppose it does arise by integration; what would the formula – where would I be led? Where would I be led? So I suppose that everything works out as smoothly as possible, what are the consequences of that? All right. So let's say that the Fourier transform paired with is the integral for minus infinity to infinity of a Fourier transform of t of x, of x, dx. All right. So, again, this would not be necessarily the case in general because distribution is not always given by integration. But suppose it is. And what has to happen? All right. So suppose – and again – I'm supposing that everything nice is happening, so the Fourier transform is given by an integral, it's the integral for minus infinity to infinity – e to the – I've already used x here, so let me write this – minus two pi i x y, t of y, d y, then times of x, dx. Supposing that I can do this, all right, supposing that everything is nice enough that I can do that. All right. Then – all right – you know, you buy the premise, you have to buy the gag – you have to follow your pencil all the way through and see where it leads you. So then I can combine all that – let me write this as one big integral: minus infinity to infinity minus infinity to infinity, then I'm gonna split it up again. So e to the minus two pi i x y, t of y, of x, dx, dy. Or I guess I wrote it dy, dx – dy dx – doesn't matter because now I'm gonna split everything apart again. Dy dx. All right. So now I'm gonna swap the order of integration, and I'm gonna put the – did I call it or ? Sorry. This is ; my apologies; and this is , okay. So now I'm gonna swap the order of integration. I'm gonna put the with the complex exponential and leave the t alone. Alone.

And write this as the integral for minus infinity to infinity, the interval for minus infinity to infinity – e to the minus two pi i x y, of x, dx, and then the result is integrated against t of y, dy. Now, what is inside the integral? Everything is as nice as it could possibly be, he said bouncing up and down. So this is the integral, this is the Fourier transform of , evaluated at y because I'm integrating e to the minus two pi x y, of x dx, so this is the integral from minus infinity to infinity of the Fourier transform of at y times t of y, y. Okay. Where do we start? Where do we finish? We started with – supposing everything was as nice as it could be, and the pairing was given by integration, we had this expression: the Fourier transform of t paired with , is the integral for minus infinity to infinity of t of y. The Fourier transform of y dy, and you have to look at this and look at this backwards in the sense that what appears here is the pairing of t and the Fourier transform with – of . All right. If everything is as nice as possible, and the pairing is given by integration, then what has occurred by our following our trusty pencil is that this is the pairing of t with the Fourier transform of . Okay. So now you say to yourself, 'If everything were as nice as possible, how would I pair of t with – what would the Fourier transform of t be paired with ?' It would be the same as if t were paired with the Fourier transform of . Okay. All right. Now, this right-hand side is gonna make sense regardless of what I did before I got to it. All right. Why? Because t is a tempered distribution; t operates on Schwartz functions. T operates on rapidly decreasing function. If is a rapidly decreasing function, so is it's Fourier transform. It's the property of – that's the property of Schwartz functions; that's the property of rapidly decreasing functions. So it makes sense for t to operate on this. All right. This left-hand side may not have initially made sense, but this right hand side will make sense. All right. Now, bless your souls. What is the thing to do? The thing to do is to turn the solution of the problem into a definition. All right. Turn this into a definition. Define the Fourier transform. So again, I'm telling you – so if we're given distribution t – tempered distribution t, so t is tempered distribution. I wanna define the Fourier transform; I wanna define a distribution. You give me a test function; I have to tell you how the distribution operates on that test function. I wanna define the Fourier transform of t by how it operates on a test function. The Fourier transform of t operating on is, by definition, t operating on the Fourier transform of .

On the right-hand side, this is the classical Fourier transform of , given by an integral because is as nice a function as it can be. This is a new , so to speak. This is a definition of a Fourier transform of the tempered distribution t. How do I define the distribution? You give me a test function; I have to tell you how it operates. It operates by t operating on the Fourier transform. Now, the right-hand side makes sense because if is a rapidly decreasing function, then so is of . I could smoke a quiet cigarette here, but I don't smoke. It's so cool. Now, look. You might say, 'What a cheat.' I mean, what a cheat. I mean, you're telling me after all this that the Fourier transform of t operating on is t operating on the Fourier transform? That's what I'm telling you, and that's because I'm compelled to tell you that, all right? It has to work out that way if everything else is gonna be consistent. All right. Now, how should we define the inverse Fourier transform? If we define the Fourier transform, how shall we define the inverse Fourier transform of a distribution? Well, the inverse Fourier transform of t operating on must be nothing other than t operating on the inverse Fourier transform of . Okay. What else is it gonna be?

And now, let's proof Fourier inversion. Fourier inversion would say the inverse Fourier transform of the Fourier transform of t is equal to t for any distribution. And also the Fourier transform of the inverse Fourier transform of t is equal to t. All right. It's the most important theorem in the field. All right. If you can't invert the transform, it's not gonna do any good practically, so this is the most important theorem, and it's trivial. It's an absolute triviality. Why? Because everything has been carefully defined. It is a triviality because all the terms in this expression, and how to operate with them have been precisely defined. What is the inverse Fourier transform of the Fourier transform t operating on ? The inverse Fourier transform of something is equal to the something – Fourier transform – applied to the inverse Fourier transform of . What is the Fourier transform of t operating on this test function, which is legit because the inverse Fourier transform is again a rapidly decreasing function, it is t operating on the inverse Fourier transform – excuse me – on the – excuse me – it is t operating on the Fourier transform of the inverse Fourier transform of . Fourier transform operating on something is the something operating on the Fourier transform. Its t operating the Fourier transform of the inverse Fourier transform of . By – in the space of functions, which are as nice as possible for Fourier transforms, classical Fourier transforms, Fourier inversion works. So this is t operating on . Where did we start? Where did we finish? We have found that the inverse Fourier transform of the Fourier transform t operating on is the same thing as t operating on . For every test function , and therefore, the inverse Fourier transform of the Fourier transform t must be t. Period. The most important theorem on the subject emerges absolutely effortlessly because all the terms were properly defined. The calculation just took care of itself. All right. The effort to try to prove this in the classical case is just murder. All right. It's just murder. You can prove it in the case of Schwartz functions.

You can prove it in the case of rapidly decreasing functions, and the proof actually is not in all detail, but in most of the details is given in the notes. All right. And there it goes pretty smoothly because the functions are as nice as possible. But if you try to start to prove Fourier inversion when the functions are not quite as nice as possible, you run into all sorts of mathematical snares. All right.

But if you broaden your perspective, and that's what's really required here, if you broaden your perspective on this and let these new ideas is, then it becomes just a piece of cake. It becomes a deduction immediately from the definition. All right. Now, let's have a Fourier transform hit parade. Let's calculate some Fourier transforms because one of the things I said to you, and I want to show you now in a way I hope you'll believe, is that you can calculate with this definition. It's not just for proving theorems; it's not just for making things rigorous, it's also provides you a way of calculating in a way that you can have greater confidence in your answers. Let's find the Fourier transform of delta. All right. The mysterious delta function, I mean, trying to find the Fourier transform of that, I mean, my God, you know, what could be harder in the classical case? But when you give the setup in terms of distributions, it is so simple. Watch. To define the Fourier transform of delta, by definition, the Fourier transform of delta operating on a function , is delta operating on the Fourier transform of . That's the definition of the Fourier transform. Okay. You give me a test function; I have to tell you how it operates on the test function. The Fourier transform of delta operating on , is by definition, delta operating on the Fourier transform of .

But now, how does delta operate on a test function? Delta operates on the test function by evaluating. That's the Fourier transform of evaluated at zero. Okay. But now, what is the Fourier transform of at zero? Now I write down the classical definition of the Fourier transform and evaluate it at zero. I can do that because is just as nice a function as it can be. This is, by definition, the integral from minus infinity to infinity of either the minus two pi i zero times x, of x, dx. In other words, is the integral for minus infinity to infinity of one, e to the zero, one times of x dx. And now, you have to look at this, and have to look at it with new eyes – the integral for minus infinity to infinity of one times x is the function – the constant function one paired with if I regard the constant function one as inducing a distribution. Where do we start? Where do we finish? We found that the Fourier transform of delta paired with is the same thing as one paired with , and that is true for any test function , and therefore, the Fourier transform of delta is equal to one. The simplest of all distributions has the simplest of all Fourier transforms – the constant function – one. The Fourier transform of delta is one. Airtight. Airtight. All right. Now, you may very well have seen this in other classes. You may very well have seen other, sort of, tortured derivations of this property. If the Fourier transform of delta is one, but this is the right derivation of this property. Okay. Nothing is in question here, absolutely nothing. Now, you see how easily I can compute that? All right. I've got a certain amount of practice. It's true. But none of these steps was hard, and it's worth your effort to, sort of, go through and say these things to yourself out loud. Say them out loud, all right? Because it works so nicely and so easily.

By the way, this is another example of – as a matter of fact, this may be the extreme example of the sort of dual relationship between concentrated in one domain and spread out in the other domain. All right. Delta is infinitely concentrated. All right. Sort of by definition, that's what delta is supposed to be. Delta is supposed to be the limit of functions that are concentrating at a point. Its Fourier transform, one, is uniformly spread out. Isn't that nice? That's the other thing you have to sort of get – say to yourself – every time you have one, I mean, I know it's hard because there are so many little bits and pieces like this. But that's one of the things you start to get used to when you work in this subject for awhile, when you use it, is you start to have these little checks, you know? These little interpretations that you carry with you from old situations to new situations. The old situation was in the stretch theorem, concentration in the time domain means spreading out in the frequency domain, and vice versa. All right. So here, concentration of the time domain is spread out in the frequency domain in the most uniform and the most extreme case. Let's do some more.

What is the Fourier transform of delta sub-a, the shifted delta function? All right. What about the shifted delta function, delta sub-a? All right. What is the Fourier transform? The Fourier transform of delta sub-a, I have to tell you how it operates on a test function. The Fourier transform of delta sub-a is, by definition, delta sub-a operating the Fourier transform of . But why does delta sub-a operating on anything, it evaluates the thing at a. That is, this is the Fourier transform of at a. All right. What is the Fourier transform of at a? It is the integral for minus infinity to infinity, d to the minus two pi i a x, of x, dx. And now, again you have to look at this with new eyes. And you have to say to yourself, 'This is the pairing of the complex exponential with .' This is the pairing of either the minus two pi i a x with . Where do we start? Where do we finish? We have the Fourier transform of delta sub-a paired with is equal to, either the minus two pi i a x paired with , and therefore, that holds for every test function that identifies the Fourier transform of delta sub-a as e to the minus two pi i a x. Where the right-hand side is understood as a distribution. Okay. It's the distribution which operates on a test function by this pairing. You may have seen this, you may have seen some tortured derivation of this, too, but this is the right derivation of it. And while we're at it, why don't we find the Fourier transform of the exponential – complex exponential? Hey, hey, hey. Let's find the Fourier transform of – where is it here – yes – let's find the Fourier transform of e to the two pi i a x. How do I do that? Where this is understood as a distribution. This would not exist in the classical case, right? I mean, you can't find the Fourier transform just by computing the integral, but we can by pairing.

The Fourier transform would be the two pi i a x paired with is equal to, by definition, either the two pi i a x paired with the Fourier transform of . That is the integral for minus infinity to infinity of – this is the function pairs with Fourier transform either the two pi i a x paired with the Fourier transform of x dx. Now, look at that and what do you see? What do you see? What do you see? You see the inverse Fourier – this is either the two pi i a x times the Fourier transform of , classical Fourier transform. You are computing there the inverse Fourier transform evaluated at a. This is of a because this is the classical inverse Fourier transform of the Fourier transform at a. So it is of a. But of a is the delta function at a paired with . If you use your new eyes, all right? So, where do we start?

Where do we finish? We started with that, we finished with that. The Fourier transform of either the two pi i a x paired with is the same thing as delta sub-a paired with . That identifies the Fourier transform at either two pi i a x as delta sub-a. Airtight. No question. Take the particular case – actually, when a is equal to zero. Take the case when a is equal to zero. If a is equal to zero, I have the Fourier transform of one. Either the two pi i zero – and what do I get? The Fourier transform of one is delta sub-zero. The Fourier transform of delta is one; the Fourier transform of one is delta. No doubt about it. No doubt about it.

And again, this is a nice illustration of spread out in the time domain, concentrated in the frequency domain – uniformly spread out in time infinitely concentrated in frequency. Delta infinitely concentrated, all right? It works. It works great. A couple of more, couple of more. How about – do it up here. Signs and cosigns. Signs and cosigns do not have classical Fourier transforms, but they have generalized Fourier transforms because signs and cosigns make sense as distributions. All right. For instance, cosign of two pi a x – I'll stick with the sort of scaling by a. All right. That's easy to do because that's one-half, e to the two pi i a x plus e to the minus two pi i a x. And I take the Fourier transform of each part – each piece. For the Fourier transform of cosign of two pi a x is one-half the Fourier transform of this, which is delta sub-a plus the Fourier transform of this, which is delta minus a, or plus a by minus a in this expression, so it's delta sub-minus a. Okay. So simple. So simple. Really.

There's a graphical way of representing that. The graphical picture usually goes like this. I write it by two spikes, one at a and one at minus a, here's zero. So this is usually – people usually denote the Fourier transform graphically by this. And that's fine, you know, that's fine, you can write it like that. The picture is that. How about the sign – the sign function is just as easy because the sign function can also be expressed in terms of complex exponentials, the imaginary part of the complex exponential. So the sign of two pi x – two pi a x is two pi a x is one over two i times e to the two pi i a x minus e to the minus two pi i a x. And so it's Fourier transform, which I guess I'll do over here, is the difference of two delta functions times this complex number, one over two i. The Fourier transform of sign of two pi a x is one over two i times delta sub-a minus delta sub-minus a. Nothing to it. Nothing to it. All right.

Once again, I said a little while ago that the problem with a classical Fourier transform is that it didn't make sense on the functions that you really want it to make sense on, like, the functions that society needs: trig functions, constant functions, and so on. All right. But now, if you open your mind and open your eyes, all those things work. All those things work and it's effortless. It's effortless. Now, it takes some effort before it becomes this effortless. Yeah?

Student: Do we have this more general framework; is that also how you define a Fourier transform of classical functions? That you could do before, like the rec functions?

Instructor (Brad Osgood) All right. So that's a good question. The question is what happens to all that we did before? Now, honestly, to be the most honest and rigorous about this, you should always consider tempered distribution. So that is to say, if you want to think the Fourier transform of the rec function – all right – there's two ways of doing it. You can consider it classically, that's fine. There's nothing wrong with that. Or you can say the rec function induces a distribution, right? And its Fourier transform is the sinc function also understood as a distribution. Now, that's going a little too far in some sense, in that, you know, when these functions are nice enough – when a function induces a distribution, you tend to blur the distinction between the function and the distribution that it induces. All right. But the most proper way of saying it is you only consider the Fourier transform for tempered distributions. That's where it's properly defined, and so classical functions where we computed the Fourier transform before, should be understood as distributions. And everything should be in that framework, all right? But then you have to say to yourself, 'Look, we're not kids here. All right. I think we have to take a reasonable approach to this.' And you don't want to abandon what you have done in the past. All right. So it's one of these things where you have to keep a little tension, a little cognitive dissonance in your head between the Fourier transform of tempered distributions and the Fourier transform of a function when it does make sense. All right. So – and I'm going to blur that line. All right. That is, I'm going to write the Fourier transform of the rec function is the sinc function without worrying about it, or without saying, really it has to be distributions – it has to be understood in the sense of distributions and so on.

So, for example, when you write, now, and matter of fact – and I say this in the book – you can finally justify completely that the Fourier transform of the sinc function is the rec function; or the inverse Fourier transform of the sinc function is the rec function. Same thing. All right. You can do dual; you can do all those things. The proper way of understanding this is in the sense of distributions because the Fourier transform of the sinc function exists – sinc induces a distribution; it's a fine function as far as integrating against a Schwartz function goes. All right. So the Fourier transform of the sinc function is the distribution, this has to be understood in the sense that equality of distributions, not in terms of, sort of, [inaudible] functions. But that gets a little extreme. All right. That gets a little extreme. You know? Even religion has its limits. All right. Now, I'll say one more thing. Like I said, I just wanted you to enjoy the ride a little bit. There's a little bit more of the ride to go. Next time I'm gonna talk about derivatives of distributions, how every distribution is differentiable. We're moving that flaw from the classical theory of calculus where not every function is differentiable. Every distribution is differentiable and we're gonna get some formulas out of that.

You know, why do we go through this? You know, again, you have probably seen these things derived in some tortured way in previous classes, all right? Well, like I say, things have moved on. This is the modern way of viewing Fourier transforms and how they work. And you should, as part of your education, want to know what the modern viewpoint is. You could build a radio out of vacuum tubes, but we don't teach that anymore. All right. We don't teach vacuum tube technology. All right. You should want to know, even if only to be – only to have sort of an acquaintance that is possible to do, you should want to know how the modern view of this is. And furthermore, as they say, it's not just that it's a good point of view, you can actually compute with it with confidence. I derived effortlessly these formulas. Like, the Fourier transform of one is delta; the Fourier transform of delta is one. The derivations of those in the classical case are really quite involve and not at all, I think to my mind, convincing. All right. They were – they get you the correct answer, but, you know, the cost of torturing things to do that is quite high. All right. I see people wanting to get in. We're gonna take a break. I know there was a question in the back; you can ask me when we're on the way out, okay? All right. See everybody on Wednesday.

[End of Audio]

Duration: 51 minutes