TheFourierTransformAndItsApplications-Lecture30

Instructor (Brad Osgood):And let's see, review so – oh, I'm on. Man, give me a chance here, will you? Where is the review session? The usual place? Review session today – what time?

Student:[Inaudible].

Instructor (Brad Osgood):4:15 – and where is it?

Student: Skilling 191.

Instructor (Brad Osgood):Skilling 191. Okay. All right. Let me call your attention to the various announcements on the board, some of which you have seen before. So the final exam is a week from yesterday, next Thursday [inaudible] 11:30 in the morning, Dinkelspiel Auditorium. Be there or be dead. Now, I know there are a couple people who have conflicts, so I'm working out some times with them. Once again, it's open notes, open book. We'll provide the blue books. We'll provide the formula sheets. You provide the knowledge, and the answer, and the correct answers. Please make sure your answers are correct. It makes the exams much easier to grade. Let's see. Any questions on that? Watch the website for postings of various things. I'm not sure if the solutions for last year's final are posted yet or not. Actually – you don't have them? Anyway, if they're – I posted the final from last year. I'm also gonna post the solutions if it hasn't been done yet, so that'll be up there, and there'll be other things, too. I think I owe you some homework solutions and things like that that will be up, and other sorts of announcements. So watch the webpage. Watch your e-mail for other exciting announcements. There is a review session today – the usual review session, I guess, at 4:15 in Skilling 191, and also the TAs will have the regular office hours next week. I'm gonna have to check on my office hours because I know I have some meetings and things like that that I have to go to, so I have to let you know on that. So again, watch the webpage for other sorts of announcements.

Also, you should have gotten an announcement from the registrar's office that the online teaching evaluation form is open, has been open, and will be open through next week, I guess, so I urge you please to go to whatever the appropriate site is on Axess. Don't leave this room. Where do you think you're going? Oh, too late. Go to the – sit down. All right. Go to the appropriate website on Axess and please fill out your teaching evaluation. Now, every time I teach a course at the end of the quarter, as a public service to the students in the class, I offer a few tips on how to fill out the teaching evaluations because I know you're very busy, and sometimes stressed, and sometimes are stuck for something to say, so let me give you some sample adjectives you might wanna use, like for instance, brilliant, dazzling, never have I had a richer more fulfilling intellectual experience, and of course, close-up please, handsome. All right. Okay. Dolly back. So now I wanna finish things up with a treatment of inverting the Radon transform, so that's how I wanna finish things up. And again, I'm sorry that we don't have a little bit more time to do a little bit more detail and go into the finer points of this because it's really – and show you actually some of the ways that it's implemented because it's interesting, but there are plenty of ways of following this up, certainly in courses on medical imaging and so on with any – and as you know, I'm sure we have a very active medical imaging group in the department.

So this is tomography and inverting the Radon transform. So I'll remind you of the setup, what the Radon transform is, and why this is an interesting problem, and why the question is formulated this way. So the setup is we have this two-dimensional slice full of gunk, 2-D region – let's just put it delicately – of variable density, density mu of X1 X2. And the idea is that if you can find mu, if you can describe – you don't know mu, but it's variable and the question is if you can reconstruct mu then you can tell what's inside. So you pass X-rays through the region, and you measure the integral of mu along various lines. So you're restricting mu to different lines, and that's actually what gets measured. You think of that as the intensity of the X-ray, or related to the intensity of the X-ray as it exits the region. So it enters with a certain intensity that you know. It gets scattered, it gets attenuated by all the junk that's in there, and that's what you measure. And from that, what you're measuring actually is the integral of L along mu. So you restrict mu to a line, you integrate it, and that's what you're measuring. So this is along various lines, along line L through the region. So you measure that.

And the question is if we know this, these values, along all lines, many lines, whatever – along lines through the region, can we get mu? Can you reconstruct mu by knowing those integrals? Now you think about this as a transform question, and already that's not so obvious. That's not an obvious step. As a matter of fact, none of this is obvious. It took I think a lot of insight, intuition, luck, whatever to formulate the problem this way and actually figure out the solution because as I say, when you see how this is solved, it's just gobsmackingly amazing. So you think of this as a transform problem. And what I mean by that is with mu fixed so it's unknown, we have a correspondence of lines to numbers given by this formula. That's a line, and the integral gives you a number. And that correspondence you think of as a transform of mu evaluated on a line L. That is to say – and I write it R because it's the Radon transform – the Radon transform of mu evaluated on a line L is exactly that formula, the integral along L of mu, so it's called the Radon transform.

It was introduced – I actually knew the history of this a little bit more thoroughly, and I cannot recall it now. It was certainly not introduced in the context of X-ray tomography or anything else. It was introduced for purely mathematical reasons, for interesting geometric reasons. The idea was to sort of study the geometry of a region by knowing integrals of sections through it just as a purely mathematical question. I don't think there were any practical implications that were anticipated or attempted certainly at the time it was introduced. So our question is – so you know all these values. All right? You know all these values, and the question is can you invert the transform? Can you find mu, given that you know all the values of its transform? Knowing all values, R mu of L – can we invert – or another way of putting it is can we invert R? Okay. Now, this looks pretty abstract, so to make this tractable I have to introduce coordinates. All right? I wanna write things in coordinates. And what I mean by that is not coordinates on the plane, not something replacing the X1 and X2 coordinates, but I want a coordinate description of the line, and actually more precisely what I want is I want a way of coordinatizing all possible lines. I want a coordinate description of the family of all lines. I wanna write this – I wanna write R mu L in coordinates, so script R.

What I mean by that is I want coordinates that describe family of the lines L in the plane. Now that's actually not outlandish at all. And in fact, you've actually seen possible coordinates when you first learned about writing equations of lines, so let me give you an example, not the example that we're ultimately gonna use, but an example that you're very familiar with, and that's what we call at least in America the slope intercept form for the equation of a line, so e.g. for example, you learned to write – You can write equation of a line as Y equals MX plus B. That's one of the first things you learned. And you know that M is the slope and B is the intercept – B and slope M. Now you can think of the pair M and B as giving the coordinates of a line. If I specify M and I specify B, that determines the line. So as the pairs M and B vary, I describe lines in the plane. So MB gives coordinates for describing a line. It describes a line, gives the line. Let me just say describes the line, the unique line Y equals MX plus B. So that's a set of parameters. That's a set of coordinates that describe the lines in the plane. It's not the best one to use certainly for our problem, and there's a problem with this set of coordinates because it omits – it doesn't give you a description of the vertical lines of infinite slope. So it's not the best – not a good set of coordinates because it'll miss the vertical lines. But certainly, as M and B vary, M can vary between zero and infinity, can equal to zero but not equal to infinity – zero is horizontal lines. Infinity is vertical lines, so that's a problem. And B can vary between minus infinity and infinity. As M and B vary over that range, you're describing all the possible lines through the origin, which should be all the possible lines in the plane, except the vertical lines. So this omits – so here again, minus infinity less than B less than infinity, and M bigger than or equal to zero less than infinity describes all but the vertical lines.

So that's not so good. You can sort of fool around with it a little bit and try to make it better, but for our problem there's a better set of coordinates. It shouldn't surprise you that some set of coordinates to a particular problem than another set of coordinates. Polar coordinates is better suited to describing problems when there's circular symmetry because the equations are simpler in polar coordinates when there's circular symmetry than they might be in Cartesian coordinates. So you wanna choose the coordinates that make the equations as simple as possible, or somehow that reveal the essential structure or symmetry of the problem, and do so in a way that's helpful for the calculation. All right. So for us actually, although it may not be immediately – it may not be evidence from what I've said, when you pass lines through the region, certainly in applications, when you're sending X-rays through, you tend to send X-rays through along parallel lines. So for us, a natural configuration of lines in the problem is something like this. Here's the region, and you may wanna pass through a bunch of – you don't send a single X-ray through, but you send a bunch of X-rays through along parallel lines, all making the same angle but parallel lines. And then you change the angle and send another bunch of parallel lines through, something that looks like this. Okay? So a family of parallel lines.

And is there a coordinate description of the family of all lines that makes it easy to write down lines like that? Well, yes. And it's actually – if you phrase it this way, it's not so unnatural. As a matter of fact, I'd like to think it's natural, although everything looks a little unnatural when you first see it. What is common to these lines? The thing that's common to these lines is they all have the same normal vector, or they make the same angle with the horizontal axis, say something like that. So that's gonna be one of our parameters is the parameter that describes the orientation of the normal vector. As a matter of fact, let's start with a single line and describe a single line in a way that's gonna allow me to easily describe families of lines like this. So for a single line, I could take something like – as a matter of fact, let me take a single line through the origin. As a matter of fact, let me make it look like it goes through the origin. Make a single line through the origin, all right? Like so. And that line is determined by its normal vector, or what is the same thing? It's determined by the angle that the normal vector – I'll call it phi – that the normal vector makes with the X1 axis. So I consider I fixed the two axes X1 and X2, and then the line's determined by phi.

The normal vector which I'm gonna need, so let me right it down – as a matter of fact, I'll take the unit normal vector is just cosine of phi sine of phi. That's a vector of length one that's perpendicular to the line. The unit normal vector is say N is cosine of phi sine of phi, but I don't need two numbers. I don't need the cosine and the sine to describe it. I just need the angle phi. That describes it. And here, what is the range of phi? Phi should vary from zero to pi, and you strip the inequality at least on one of the sides, let's say on the right-hand side. Phi equals zero means the line is vertical. It means it's making an angle of zero with the horizontal axis, so that's straight up. And then as phi moves, the line rotates, and when phi is equal to zero it's vertical. When phi is equal to pi, it's also vertical, but you don't wanna have the same line described twice, so you have a strict inequality on this side. So phi equals zero describes a vertical line. Phi equals pi also describes a vertical line. You don't wanna describe the same line twice, so you have a strict inequality here.

So that's a single parameter that describes all lines through the origin. How do I get other lines? Well, I take a line with the same normal vector – and again, I'm interested actually – I have in mind later on I'm interested in families of parallel lines, so I take a line with the same normal vector, and the other parameter that determines it uniquely is its distance from the origin, so a line not through the origin. The picture would be something like this. It has the same – here's the line through the origin with that normal vector. So here's the same normal vector. Here's the angle phi. But there's also a distance rho from the origin. I'll call it rho. So it's determined by phi and a distance rho from the origin, and that determines the line almost. I'll say a little bit more here. There's a slight subtle point here because I don't just wanna take the distance. How do I distinguish between that line – we draw two lines, and I wanna be able to distinguish between the two. Here's one line, and here's a parallel line. They're the same distance from the origin. Here's a normal vector, and here's a normal vector. The normal vector's unambiguous because I decide that phi goes with – the direction of the normal vector's unambiguous because I decide goes between zero and pi, so if the normal vector – if this line goes that way, the normal vector for this line also goes that way. They're both the same distance from the origin.

How do I distinguish between the two? I don't consider just the distance, but I consider the sine to distance. So I consider the sine distance.

So what I mean by that is rho is positive if I get from the origin to the line in the direction of the normal vector, and rho is negative if I get from zero to the line in the direction opposite the normal vector. Rho is positive if I get from the origin to the line in the direction of the normal vector. So that would be the case here. The normal vector's going that way. I get from the origin to the line by moving in the direction of the normal vector. Here rho is negative because I go the other way. Here rho is positive. Here rho is negative. So rho is just – I'll write it down. Rho is negative. Rho is equal to zero of course if I have a line for the origin. So rho is negative if I move – if I get from origin to a line by moving in the direction opposite to the normal vector, like this case. So rho can vary between minus infinity and infinity, so the range of my parameters, the range of my coordinates for the lines is – and I get all lines this way – is zero less than or equal to phi less than pi and minus infinity less than or equal to – less than rho less than plus infinity. I describe all the lines in the plane by those two coordinates. So a give rho and a given phi tell me a given line, specify a given line.

And you'll notice that actually families of parallel lines are described very nicely in this coordinate system because a family of parallel lines means phi is fixed and rho varies. So a set of lines like this – something like that or at a different angle would be phi fixed, rho varies. Okay? That's nice. Now how do you write the equation of a line? I wanna write the – transforms are set up generally speaking – and we'll see actually. We're gonna bring the Fourier transform into this in a way that's just unbelievable – in Cartesian form, so I wanna write a Cartesian equation of the line in these coordinates – of a particular line. So what is the Cartesian equation of the line specified by a given pair rho, phi? Notice here I have a distance and I have an angle, but these aren't polar coordinates. These aren't polar coordinates in the plane. These are line coordinates. These are coordinates of lines. Well, that's not hard to see, and I will refer you to the notes for a little bit more of a derivation of this. Let me just say what the formula is, and again it's something not actually so different from what you've seen before most likely, but maybe not written in quite the same terminology or quite the same language. It's given by in vector form X dot N equals rho, very simple. N is equal to cosine phi sine phi. That's the Cartesian equation in vector form of a line which is given by a given normal direction, a given unit normal, and a distance rho from the origin, sine distance plus or minus.

And if I write that on coordinates, it would be cosine of phi times X1 plus sine of phi times X2 equals rho. That's the equation. Now there's more magic. And again here, unfortunately I don't have time to derive this for you, and I'm gonna see – I have a set of notes on this, but it gets a little involved. Not so bad, it's quite interesting. It's another example of how things in higher dimensions can be a little richer. The variety of objects you find at higher dimensions can be richer than in one dimension, and this has to do with evaluating a line integral via a delta function. So I wanna use this to evaluate an integral like this – an integral a function of two variables, X1, X2, along a line via a delta function. What you do is you consider – so here I just have to state a fact for you. And then with this, we are set to go, we are good to go for the rest of our trip. You consider – I hate that, like a mathematical weasel word, “consider.” Like what could be more natural in the world to consider the following ridiculously complicated looking object. Consider a delta function along the line. You can consider a delta function along the line described this way, so describe the way I wrote it. Let me write it like this: rho minus X1 cosine phi minus X2 sine phi equals zero. That is to say delta of rho minus those things, X1 cosine phi minus X2 sine phi.

Now you can think of this – so this is so-called line impulse. So you can think of this intuitively as a delta function which is concentrated along the line, so it’s infinite – we don’t think these ways because we’re very sophisticated. It’s a distribution, blah blah blah, but if you wanna think about it easily, think about it as concentrated along the line. So it’s infinite along the line, zero off the line, and it has a special property with respect to integration. So it’s infinite along the line, zero off the line, and the key property is for us is what it does under integration. It’s analogous to the one-dimensional case, but it’s perfectly suited to our problem. So zero off the line like all good deltas, infinity on the line – I can write this shamelessly, right? And for the integral what happens is – I’ll give this a board of its own. This is really a key step because it allow – well, you’ll see. One thing at a time. The line integral of mu along the line – so imagine mu is – when you’re integrating mu along the line, it’s as if you want to concentrate mu along the line or just restrict mu to the line. That’s given by integrating over the whole plane mu against the delta function concentrated on the line. This is mu of X1 X2 integrated against the delta function of the line or the line impulse associated with the – the line impulse for the line. X – rho minus X1 cosine phi minus X2 sine phi VX1 VX2. Now that’s a fact that I’m not gonna prove for you. You have to know a little bit more carefully how line impulses are defined and so on, but it is analogous to how delta functions work in connection with integration if you think of them in terms of integration with the one-dimensional case, or you integrate a function against the delta function – it consecrates the function at a point.

Here the delta function is concentrated along a line. To integrate a function against the delta function concentrated on a line, it sort of drops at one dimension so to speak and integrates the function along the line. You haven’t seen this formula, but it is analogous to the kind of formulas you have seen before, and if you really want, I’ll post a set of notes on line impulses. I haven’t had a chance to incorporate them into the notes because they need a little work, but it’ll give you a rough idea if you really wanna know, but it’s dangerous stuff. Watch this. We are set. I’m gonna invert the Radon transform. It is unbelievable. So again here’s the setup. I’m imagining myself – although you won’t necessarily see it in the calculation, I’m imagining that I’m fixing phi and I’m letting rho vary. You wanna use this to invert the Radon transform mu. So again, mu is this unknown function. I know the values of R of mu because I know the values of the line interval. Remember, R of mu – I’ll write it down one more time. R of mu – I could even write it in – R of mu in coordinates would be – say rho phi would be the integral of the line specified by rho and phi of mu. It’s the integral along the line. And I know how to concentrate that. I know I can write this out. I can write this as – that line integral can be written in turn as a double integral – let me write the whole thing out here.

The integral from minus infinity to infinity of mu X1 X2 delta rho minus X1 cosine phi minus X2 sine phi DX1 DX2. Great, so I've now replaced a simple looking formula by a complicated looking formula, and if you like that, you're gonna love what's gonna happen now. Just follow along for the ride. Enjoy it. Don't even take notes. Just sit back. Relax. The hostesses will be by soon to serve nuts and drinks. Now think about phi fixed to let rho vary. Think effectively about passing a family of parallel lines. So think of phi fixed and rho varying. So think about effectively computing this thing along a whole bunch of parallel lines. So you're computing all these values as a function of rho. So with phi fixed and rho varying, you think about R mu rho phi as a function. Think of – as a function of rho with phi fixed. Now I'm going to before your very eyes take the one-dimensional Fourier transform of this function with respect to rho. Like, of course. Like what else would you do? I will take the 1-D Fourier transform of this – that function with rho varying – with respect to rho. In other words, I wanna computer the integral from minus – let's call it something like this. Let's call it the Fourier transform in the rho variable of R mu rho phi.

Now I need a dual variable, right? Because the Fourier transform always has a dual variable in the space domain and the frequency domain, so let me call it R. So let me write this as the integral from minus infinity to infinity of E to the minus two pi I R rho. Let me write it up here so it's – let me write it bigger. Phi just tags along for the ride. As a matter of fact, I'd better write this a little bit more carefully already. It's hard to write. We're gonna get caught a little bit here in problems of notation and problems of variables. It's the usual thing in a subject that you know – there are problems here with variables. There are problems with naming your variables. I'm gonna take the Fourier transform. Phi is fixed. Rho is varying. I'm gonna take the Fourier transform with respect to Rho, and I need to say what the variable – I need to give the variable in the spatial domain a name, so I'm gonna call it R. So that would be like the integral from minus infinity to infinity E to the minus two pi I R rho, Radon transform of mu at rho and phi integrated with respect to rho. Then what pops out is a function of R and phi because phi just tags along for the ride. So phi is fixed, but that's an additional parameter that's entering into the definition here, but I'm integrating with respect to rho. So what pops out is a function of R. Think of it as a function of R, but then I say that phi is sort of tagging along for the ride.

All right. Now be not afraid. Write this out. Be not afraid. Be of good cheer. The holidays are close. What is this? This is the integral from minus infinity to infinity E to the minus two pi I R rho – oh, of this double integral minus infinity to infinity, minus infinity to infinity of mu X1 X2 delta X – delta rho minus X1 cosine of phi minus X2 sine of phi DX1 DX2, and then – that's the R. And then the whole thing is integrated with respect to rho. That's just rewriting this equation substituting what I had before over there writing the line integral of mu in terms of the integral against the line impulse, the integral against the delta function. Okay? Be not afraid. Be not afraid. Flip the integrals. Do all sorts of unnatural acts. Write this as the integral from minus infinity to infinity, minus infinity to infinity mu of X1 X2 – I wanna bring the other stuff inside – the integral from minus infinity to infinity of E to the minus two pi I R rho times delta of rho minus X1 cosine phi minus X2 sine phi. That inside stuff is integrated with respect to rho, and then the whole thing is integrated with respect to X1 and X2.

So all I did was I sort of flipped the order of integration here. Inside here, first I was integrating with respect to X1 and X2, and then integrating with respect to rho, and there was this complex exponential outside. Now I'm flipping this thing around. I'm bringing this mu X1 X2 outside. I'm putting this with that. I'm integrating with respect to rho, and then the whole thing gets integrated with respect to X1 and X2. Be not afraid. Now look at that inside integral. We've seen integrals like that before. We have. Think about this as an engineer, not as a damn mathematician. As an engineer you're integrating a function against the delta function. Integrating a function against the delta function evaluates the function where the delta – it's the shifted delta function. You're integrating a complex exponential with respect to rho against the shifted delta function rho minus – here's the shift. All right? Here's the shift. So what is that integral on the inside? The integral from minus infinity to infinity of E to the minus two pi I R rho delta rho minus X1 cosine phi minus X2 sine phi D rho is E to the minus two pi I R X1 cosine phi minus X – or plus X2 sine phi. That's it. You're integrating a function in a complex exponential against the delta function integral phi of X times delta – integral of F of X times delta X minus Y is F of Y. You know what I'm talking about here. It's the integral. It's the convolution. It's the integral of the function against the shifted delta function. So that substitutes – the shift, it's rho minus – let me write it like this. Rho minus this, so it substitutes that into the exponent. Okay?

Let me write that out a little bit more. This is E to the minus two pi I R – let me write this X1 times R cosine phi plus X2 times R sine phi. And now, I'm gonna introduce some new variables here. I'm gonna introduce sort of dual variables to X1 and X2. So introduce C1 is equal to R cosine phi. C2 is equal to R sine phi. Now these are not polar coordinates on the C1 C2 plane, but you can sort of think of them that way. [Inaudible] because R is the dual variable to rho when you're doing Fourier transform, and phi is an angle in the X1 X2 plane, so this is sort of – these are not exactly introducing polar coordinates in the C1 C2 plane, although there is an interpretation of them that way. But if I do that then that exponential becomes E to the minus two pi I X1 times C1 plus X2 times C2. Now all this was evaluating this inside integral here, the inside integral here. The integral of E to the minus two pi I R rho times this line impulse. And that emerges to be this complex exponential E to the minus two pi I X1 times C1 plus X2 times C2 when I make this change of variable.

Plug this in to my formula. Plug this in. Plug this into what? Plug that into that. This says the Fourier transform in rho of R – this Radon transform of R mu rho phi is equal to the integral from minus infinity to infinity, the integral from minus infinity to infinity mu X1 X2 – that's what was leftover here – times this integral which I just computed to be E to the minus two pi I X1 C1 plus X2 C2 DX1 DX2. What do you see? You see the two-dimensional Fourier transform of mu. This equals the Fourier transform of mu at C1 C2, 2-D Fourier transform. All right? Let's recap. Fix phi. Let rho vary. Take the 1-D Fourier transform of the corresponding Radon transform. That is take the one-dimensional Fourier transform of this function of rho phi with respect to rho. What does it produce? It produces the two-dimensional Fourier transform of mu. Now in principle, the problem is solved. You know this. You know this expression. You measure these values. This is what you're measuring. You're measuring the Radon transform, that's the line integral of mu along this whole family of lines. As rho is varying and phi is fixed, you are measuring these values. You know this function.

Because you know this function, you can compute in theory its Fourier transform. You can compute its Fourier transform with respect to rho. Now what does that tell you? Computing the Fourier transform of this thing with respect to rho gives you the two-dimensional Fourier transform of mu. That tells you you can find mu by taking the inverse two-dimensional Fourier transform of what you have found. You can now find mu, so you compute once again the Fourier transform in the rho variable of this R mu rho phi. And that gives you a function. Now you have to make the changes of variable and all this jazz. You have to substitute C1 equals R cosine phi and C2 equals R sine phi and all the rest of that stuff. Okay, but what that results in at the end of the day is a function of two variables, C1 and C2, so you make the change of variables. C1 is equal to R cosine of phi. That's known. C2 is equal to R sine of phi. That's known. So this thing results in a function of two variables.

Just wanna make sure you understand exactly why this can solve the problem and in what form. So you compute the one-dimensional Fourier transform of this. You make this change of variables. This results in a function – let's call it G – of C1 and C2. Again, you know this function. You can compute its Fourier transform. You make these changes of variables. That gives you some function that you're calling G of C1 and C2, and you know that that function is nothing other than the two-dimensional Fourier transform of mu. So you know that G of C1 C2 is the two-dimensional Fourier transform of mu of C1 C2. And you have solved your problem because now you take the inverse two-dimensional Fourier transform of this function which you have computed, and that gives you mu. Now get mu is the inverse two-dimensional Fourier transform of G. Let me just write it like that. That recovers mu by knowing all those line integrals. Done. Amazing. It's absolutely incredible. So this is the mathematical basis of CAT scans, of recovering the density function by knowing all the one-dimensional slices. Now again, I set this up this way – although the modality, although the imaging techniques in the sort of whole physics behind it is different for MR, at the end of the day actually for amazing reasons, surprising reasons I think, the same thing holds.

That is the idea of finding the – when you try to do MR imaging, you're confronted with the same sort of problem and it's solved the same way. Ultimately, you compute a two-dimensional inverse Fourier transform of like a one-dimensional Fourier transform of some auxiliary function that you introduced. It comes about in a different way, but it amounts to the same thing. It's just amazing. Now again, none of this was known when Radon introduced this idea way back in I think either the end of the 19th century or the beginning of the 20th century – maybe in the '20s, 1920s. I'm not sure when. I have to look it up again. And then found this absolutely stunning and incredibly important application. I think it's nothing short of amazing. I think it's just absolutely amazing. Now of course, to make this practical and to implement this requires a lot of work.

Everything has to be done discretely. If you're actually gonna calculate this numerically, you have to implement discrete versions of this. There are a lot of computational issues. There are very interesting issues actually with the playoff between Cartesian and polar coordinate representations of the different quantities. There are lots of things that have to be done to actually make this practical, but this is the basis for it. This mathematical derivation, this way of going from the one-dimensional Fourier transform in the one variable to the two-dimensional Fourier transform in the other variables and allowing that to solve the problem is the basis for the entire thing. And I say I think it's just absolutely remarkable.

And I think with that we are done. So I wanna thank you very much for a stimulating quarter. I hope you enjoyed it as much as I did. I think this is wonderful material. I really do think it's every day another miracle almost. And I'm sure you'll find this material as you go on in whatever you direction you go on in your studies. Whether it's electrical engineering or other different fields, you're gonna find the Fourier transform and its techniques used absolutely everywhere. And it has been my pleasure to give you what I hope has been a good introduction to it, so thank you very much.

[End of Audio]

Duration: 50 minutes